23 May 2000

To whom it may concern,

The pages contained in this package are my handwritten notes taken in the course 16.46T, Astronautical Guidance, as a graduate student in the M.I.T. Department of Aeronautics and Astronautics during the Spring Term of 1960. The lectures prior to 2 May 1960 were given by Dr. Richard Battin. The lectures starting 2 May 1960 on Relativity were given by Dr Walter Wrigley, now deceased. Dr. Battin discussed round-trip return trajectories to Venus and Mars during the lectures on 21 and 23 March 1960.

Walter M. Hollister

16.46 T ASTRONAUTICAL GUIDANCE PROF. WRIGLEY

TEXTS: STARS DR. BATTIN

STERN'S - PROBS. GRADER

Celestial Mechanics
Mathematics of Trajectories
Interplanetary Guidance

"Bodes law" 0, 3, 6, 12, 24,
Add 4 & divide by 10 to get distances of planets

Planet	Bode	Actual
Mercury	0.4	0.39
Venus	0.7	0.72
Earth	1.0	1.00
Mars	1.6	1.52
—	2.8	—
Jupiter	5.2	5.20
Saturn	10.0	9.53
Uranus	19.6	19.19
Neptune	38.8	30.07
Pluto	77.2	39.95

Kepler's Law

I. A planet moves in an elliptic orbit with the sun at one of the foci

II Rate of description of areas is constant

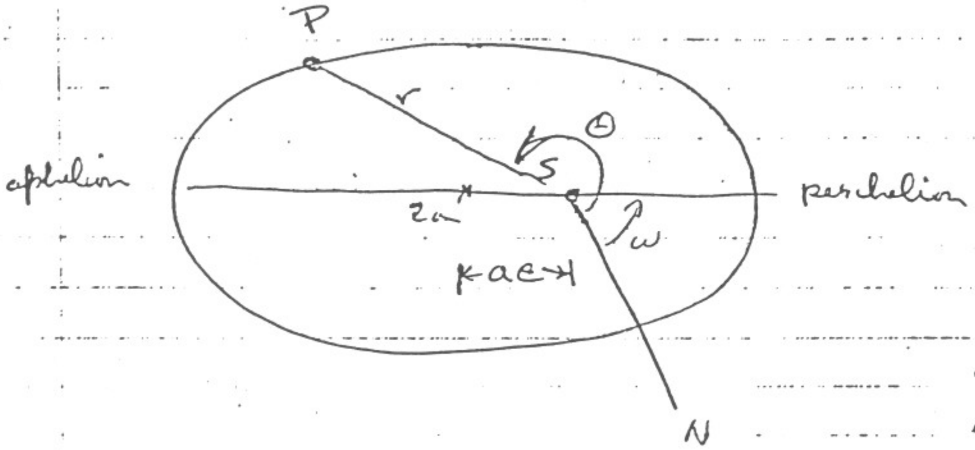
III If A is mean distance to sun & P is period

$\frac{A^3}{T^2}$ = constant is not exactly true ~~[illegible]~~ unless ~~[illegible]~~ mass sun is infinite

$$\frac{4\pi^2 a^3}{T^2} = G(M_0 + m)$$

M_0 = mass of the Sun
m = mass of planet

G = universal gravitation const



ω - longitude of perihelion
a = semi major axis
e = eccentricity

$$r = \frac{a(1-e^2)}{1+e\cos(\Theta-\omega)}$$ By Kepler's 2nd Law

$h = r^2\dot{\Theta}$ = angular momentum   T = period

$$h = r^2\dot{\Theta} = \frac{2\pi a^2\sqrt{1-e^2}}{T}$$

mean motion: $n = \frac{2\pi}{T}$ $h = na^2\sqrt{1-e^2}$

Look in Encyclopedia Britannica for measurements of G

$G = 6.66 \times 10^{-8}$ cgs units.   Commonly accepted

$G = 6.673 \pm .003 \times 10^{-8}$ Britannica

units $L^3T^{-2}M^{-1}$

$K = \sqrt{G}$ = Gaussian Constant $= 0.01720209 9$

Keep the Gaussian Constant and change the length

L – astro unit
T – mean solar day   365.2564 / year.
M – sun's mass

$\log a_1 = 0.000000013$

L ~ a.u.
T ~ year
M ~ sun's mass

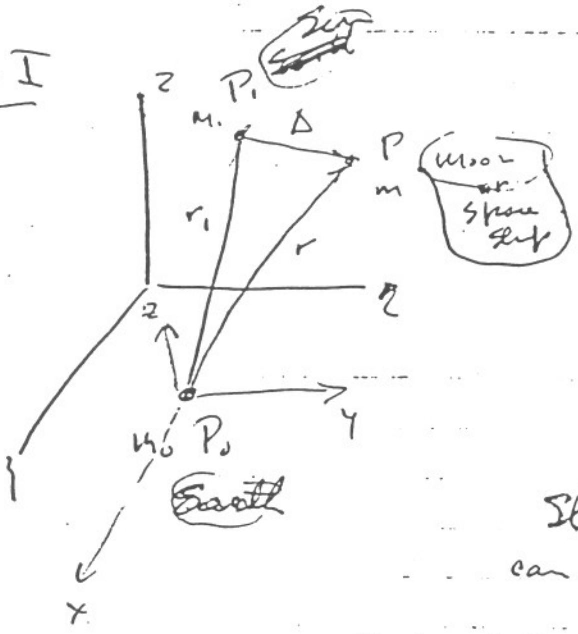
$G(m_0 + m) = \mu = 4\pi^2$ ⟵ Good for slide rule

$$F = G\frac{m_1 m_2}{r^2}$$

Gravitational Potential $\quad V = \frac{Gm}{r}$

x Comp of Force per unit mass $= \frac{\partial V}{\partial x}$

Problem I



$$\frac{d}{dt}(m\,\dot{\xi}) = m\frac{\partial V}{\partial \xi}$$

Show equations of motion can be put in the form

$$\ddot{x} + \frac{\mu x}{r^3} = \frac{\partial R}{\partial x}$$

$$\ddot{y} + \frac{\mu y}{r^3} = \frac{\partial R}{\partial y}$$

$$\ddot{z} + \frac{\mu z}{r^3} = \frac{\partial R}{\partial y}$$

$$R = Gm_1\left(\frac{1}{\Delta} - \frac{xx_1 + yy_1 + zz_1}{r_1^3}\right)$$

$\mu = G(m_0 + m)$

R = "disturbing function" due to presence of m_1

12 Feb 1960

For n-body problem

$$\ddot{x} + \frac{\mu x}{r^3} = \sum_{i=1}^{n} \frac{\partial R_i}{\partial x}$$

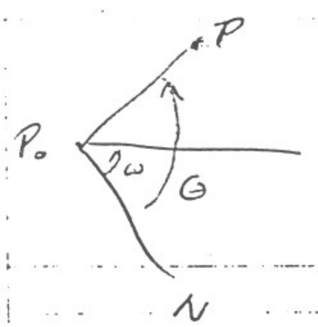
$$R_i = G m_i \left(\frac{1}{\Delta_i} - \frac{x x_i + y y_i + z z_i}{r_i^3} \right)$$

Elliptic Orbit

$$\ddot{\bar{r}} + \frac{\mu \bar{r}}{r^3} = 0$$

Can show that this motion lies in a plane

In the plane



$$* \quad \ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \qquad \boxed{r^2\dot{\theta} = h} \quad \text{Kepler's } 2^{nd} \text{ Law}$$

Let $u = \frac{1}{r}$

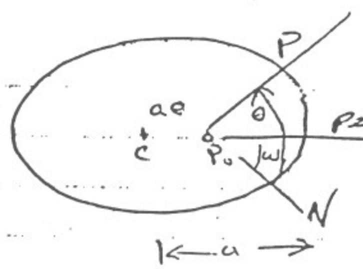
$$\frac{du}{d\theta} = \frac{du}{dt}\frac{dt}{d\theta} = -\frac{1}{r^2}\dot{r}\,\frac{r^2}{h} = -\frac{\dot{r}}{h}$$

$$\frac{d^2u}{d\theta^2} = -\frac{\ddot{r}\,r^2}{h^2}$$ & substitute for $\ddot{r}$ & $\bar{r}$ in $*$ Eq

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

$$u = \boxed{\frac{\mu}{h^2}\left[1 + e\cos(\theta - \omega)\right] = \frac{1}{r}}$$

Kepler's 1^{st} Law



$$\frac{\mu}{h^2}(1+e) = \frac{1}{a(1+e)}$$

$$h^2 = \mu a(1-e^2)$$

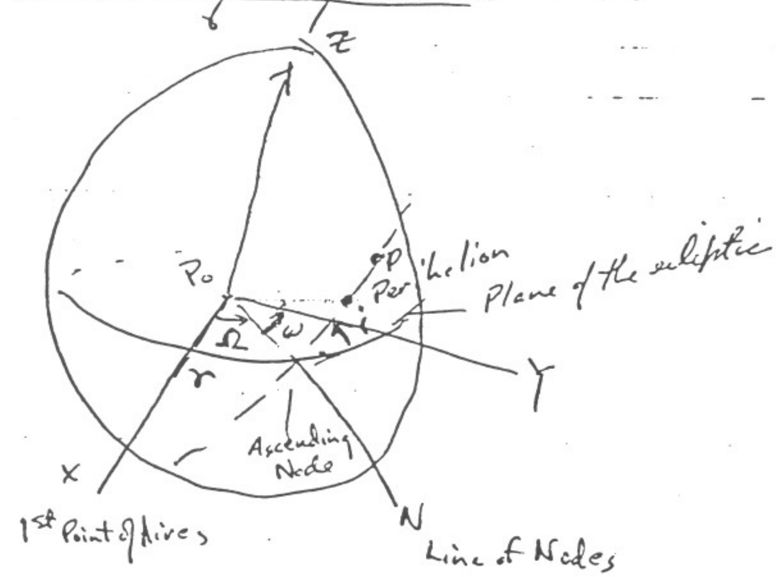
$$h = n a^2\sqrt{1-e^2}$$

$$\boxed{\mu = n^2 a^3}$$ 3rd Law

$0<e<1$ Ellipse
$e=0$ Circle
$e>1$ Hyperbola
$e=1$ Parabola

$n = \frac{2\pi}{T}$ = mean motion

6 Elements of Body Location



i - inclination angle (always +)

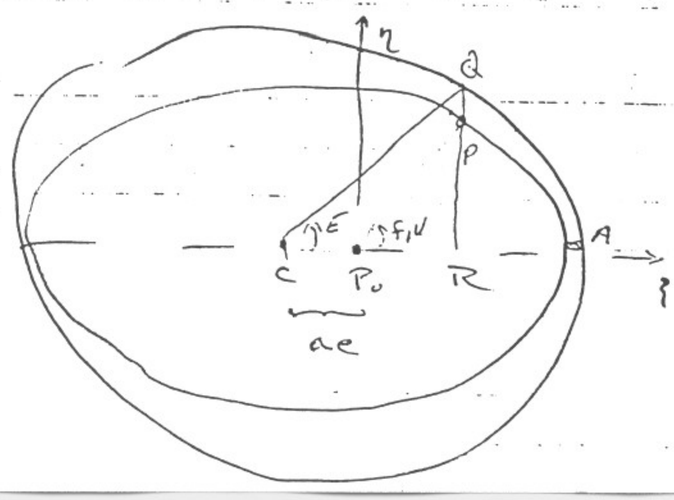
Ω - longitude of the node

τ - time of perihelion passage

ϖ = longitude of perihelion $= \Omega + \omega$ (measured in 2 separate planes)

ϖ Script π

Elements: $\Omega \quad i \quad \varpi \quad a \quad e \quad \tau$



f, v = true anomaly
E = eccentric anomaly
M = mean anomaly

$M = n(t-\tau)$

$n = \frac{2\pi}{T}$

Keplers Equation: $M = E - e \sin E$

15 Feb. 1960

$$\boxed{\begin{cases} \xi = r\cos f \\ \eta = r\sin f \end{cases} \qquad r = \frac{a(1-e^2)}{1+e\cos f}}$$

$\xi = a\cos E - ae$ from geometry

$$\frac{CR^2}{a^2} + \frac{PR^2}{b^2} = 1 \qquad \text{Ellipse}$$

$$PR^2 = \eta^2 = b^2\left(1 - \frac{CR^2}{a^2}\right) = b^2 \sin^2 E$$

$$r^2 = \xi^2 + \eta^2 = a^2\cos^2 E - 2a^2 e\cos E + a^2e^2 + b^2\sin^2 E$$

$$b^2 = a^2(1-e^2)$$

$$\boxed{\begin{aligned} r &= a(1 - e\cos E) \\ \xi &= a(\cos E - e \\ \eta &= a\sqrt{1-e^2}\,\sin E \end{aligned}}$$

Derive Kepler's Equation's using areas.

$$\frac{\text{Area } AP_0P}{t - \tau} = \frac{1}{2}h = \frac{\pi a b}{T}$$

Then $M = \frac{2}{ab}$ Area AP_0P

$$\text{Area } AP_0P = \Delta RP_0P + \Delta APR$$

$$= \frac{1}{2}a(\cos E - e)\, b\sin E + \frac{b}{a}\left(\text{Sector } ARQ - \Delta RCQ\right)$$

Put it together to get $\boxed{M = E - e\sin E}$

$$\tan\frac{f}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{E}{2}$$

References: Smart & Moulton

Techniques with Kepler's Equation

Shapiro – "Prediction of Ballistic Missile Traj. for Radar Operations" Appendix

I. E_0 = approx. solution

$E_0 + \Delta E_0$ = exact solution

$$M = E_0 + \Delta E_0 - e\sin(E_0 + \Delta E_0)$$

Assume ΔE_0 small

$\sin \Delta E_0 \sim E_0$

$\cos \Delta E_0 \sim 1$

$$\boxed{\Delta E_0 = \frac{M - M_0}{1 - e\cos E_0} \qquad M_0 = E_0 - e\sin E_0}$$

II Series Approx.

$$E = M + e\sin E$$

$$E_0 = 0$$

$$E_1 = M + e\sin E_0 = M$$

$$E_2 = M + e\sin E_1 = M + e\sin M$$

$$E_3 = M + e\sin(M + e\sin M)$$

<u>Assume e small</u>

$$E_3 = M + e\left[\sin M\cos(e\sin M) + \cos M\sin(e\sin M)\right]$$

$$= M + e\sin M + \frac{1}{2}e^2\sin 2M$$

converges for $e < 0.6$ or so

Not good for comets' etc.

III Fourier Bessel Expansion

Converges for $e < 1$

$$M = E - e\sin E$$

$$dM = (1 - e\cos E)\,dE$$

$$\frac{dE}{dM} = \frac{1}{1 - e\cos E} = A_0 + A_1\cos M + A_2\cos 2M + \cdots$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{dM}{1-e\cos E} = \frac{1}{2\pi}\int_0^{2\pi} dE = \underline{1}$$

$$A_k = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos kM}{1-e\cos E}\, dM = \frac{1}{\pi}\int_0^{\pi} \cos\left[k(E - e\sin E)\right] dE$$

Integral. def. of Bessel Function

$$= 2\, J_k(ke)$$

$$\frac{dE}{dm} = 1 + 2\sum_{k=1}^{\infty} J_k(ke)\cos km$$

$$E = M + 2\sum_{k=1}^{\infty} \frac{J_k(ke)}{k} \sin kM$$

$$J_n(x) = \frac{\left(\frac{x}{2}\right)^n}{n!}\left\{1 - \frac{\left(\frac{x}{2}\right)^2}{1(n+1)} + \frac{\left(\frac{x}{2}\right)^4}{1\cdot 2\cdot(n+1)(n+2)} - \cdots\right\}$$

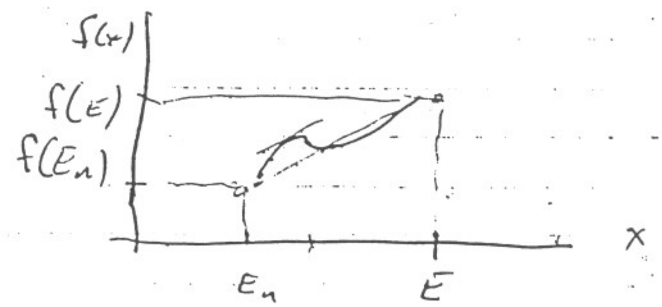
Suggested that you compare II & III out to e^3

17 Feb. 1960 Technique for computer solution of Kepler's Equation.

Series Approximation $E_{n+1} = M + e\sin(E_n) = f(E_n)$

E = correct value $E - E_{n+1}$ = error at $n+1^{th}$ step.

mean value theorem



Slope at some point on curve = Mean Slope

$$\frac{f(E_n)-f(E)}{E_n-E} = f'[E_n+\beta_n(E-E_n)] \qquad 0\le\beta_n\le 1$$

$$E_{n+1}-E = f'[E_n+\beta_n(E-E_n)](E_n-E) \qquad \text{Recursion Form.}$$

$$|f'(E)| = |e\cos E| \le e$$

$$E_{n+1}-E = (E-E_1)\prod_{n=1}^{n} f'[E_n+\beta_n(E-E_n)]$$

$$|E_{n+1}-E| \le |E|e^n \le |M+e|\,e^n$$ Establishes convergence for $e<1$

$$|E|\le M+e$$

but Kepler's Equation is only good for $e<1$

By another author

$$|E-E_{n+1}| \le M\frac{e^n}{1-e}$$

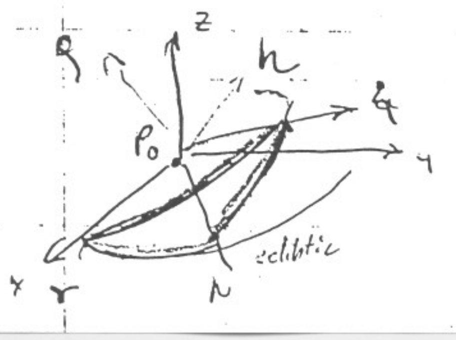
If we want $|E-E_{n+1}|<A$

$n=$ min of n_1 & n_2 $\qquad n_1 = \dfrac{\log\frac{A(1-e)}{M}}{\log e}$

$$n_2 = \frac{\log\frac{A}{M+e}}{\log e}$$

For $M=\pi$ $e=\frac{1}{2}$ $A=10^{-4}$ $n=15$ iterations

Heliocentric Inertial System

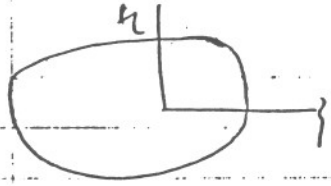


take ξ-η-ζ axes in plane of orbit
ξ through perihelion

$$\xi = a(\cos E - e)$$

$$\eta = a\sqrt{1-e^2}\,\sin E$$

$$M = E - e\sin E = n(t-\tau) \qquad n = \frac{2\pi}{T}$$



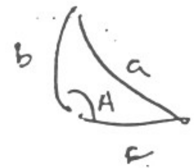
$$\vec{P} = \xi\,\vec{i}_\xi + \eta\,\vec{i}_\eta$$

$$\vec{P} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\begin{aligned} x &= l_1\xi + l_2\eta \\ y &= m_1\xi + m_2\eta \\ z &= n_1\xi + n_2\eta \end{aligned} \qquad l_1 = \bar{i}_\xi\cdot\hat{i} \quad l_2 = \bar{i}_\eta\cdot\hat{i}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l_1 & l_2 \\ m_1 & m_2 \\ n_1 & n_2 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

Consider spherical triangle



$$\cos a = \cos b\cos c + \sin b\sin c\cos A$$

In shaded triangle prev. page:

$$l_1 = \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i$$

Similarly $\quad l_2 = -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i$

$$m_1 = \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i$$

$$m_2 = -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i$$

$$n_1 = \sin\omega\sin i$$

$$n_2 = \cos\omega\sin i$$

Now have x, y, z

from a, e, Ω

$\varpi = \Omega + \omega$

i

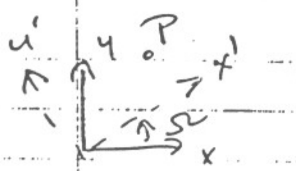
τ

Note that

$\Omega\ \omega\ \&\ i$ are

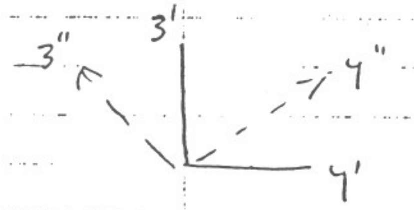
Euler angles

& 3 rotation matrices are applicable.

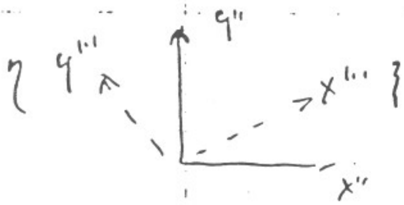


$$x = \cos\Omega\, x' - \sin\Omega\, y' \qquad z = z'$$
$$y = \sin\Omega\, x' + \cos\Omega\, y'$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$



$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$

Converts vectors from space ship coordinates int. heliocentric coord

Orthogonal Matrix ✓

19 Feb. 60

Velocity $\quad \xi = a(\cos E - e) \qquad r = a(1 - e\cos E)$

$\eta = a\sqrt{1-e^2}\,\sin E$

$$\frac{d\xi}{dt} = -a\sin E\,\frac{dE}{dt} = -\frac{a^2 n}{r}\sin E \qquad \frac{dE}{dt} = \frac{an}{r}$$

$$\frac{d\eta}{dt} = a\sqrt{1-e^2}\,\cos E\,\frac{dE}{dt} = \frac{a^2 n}{r}\sqrt{1-e^2}\,\cos E$$

$\mu = n^2 a^3$

$a^2 n = \sqrt{a\mu}$

$$\frac{d\xi}{dt} = -\frac{\sqrt{a\mu}}{r}\sin E$$

$$\frac{d\eta}{dt} = \frac{\sqrt{a\mu}}{r}\sqrt{1-e^2}\,\cos E$$

$$v^2 = \frac{a\mu}{r^2}(1 - e^2 \cos^2 E) = \frac{\mu}{r}(1 + e \cos E)$$

$$\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)} \qquad v = f(r) \quad \text{Note.}$$

Orbital Elements:

Semi major axis: a (mean distance)
Eccentricity: e
Long of the Ascending Node: Ω
Long. of perihelion: $\varpi = \Omega + \omega$
Inclination angle: i
Mean longitude at the epoch: ϵ

True longitude of the planet $\quad \theta = \Omega + \omega + f = \varpi + f$

Mean longitude of the planet $\quad = \Omega + \omega + n(t - \tau)$
$\quad = \varpi + n(t - \tau)$

Mean longitude at the epoch: $\epsilon = \varpi - n\tau$
$n\tau = \varpi - \epsilon$

$M = n(t - \tau)$

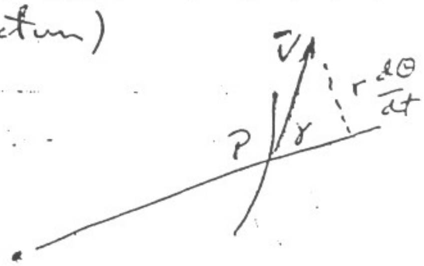
$$\boxed{M = nt - \varpi + \epsilon}$$

$$h = 2\frac{dA}{dt} = n a^2 \sqrt{1 - e^2} \qquad \mu = n^2 a^3$$

$$= \sqrt{\mu a (1 - e^2)} \qquad r = \frac{a(1 - e^2)}{1 + e\cos(\theta - \omega)}$$

$a(1 - e^2) = p$ (semi latus rectum)
(parameter)

$$h = \sqrt{\mu p}$$



$$\boxed{h = r v \sin \gamma}$$

$$r = \frac{a(1-e^2)}{1+e\cos(\theta-\omega)}$$

$$\frac{dr}{dt} = \frac{e}{p} h \sin(\theta-\omega) = V\cos\gamma$$

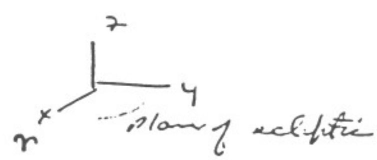
$$\begin{cases} v\cos\gamma = \mu \frac{e}{h}\sin(\theta-\omega) \\ h^2 = \mu r\left[1+e\cos(\theta-\omega)\right] \end{cases}$$

$$\boxed{e^2 = 1 + \frac{v^4 r^2 \sin^2\gamma}{\mu^2} - \frac{2v^2 r \sin^2\gamma}{\mu}}$$

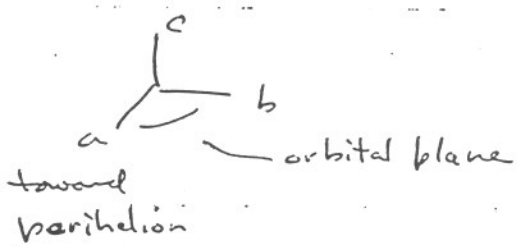
Gives e from initial conditions. — Last 2 problems in Set 3

Coordinate systems.

Ecliptic - x, y, z



Planet system - $a\,b\,c$

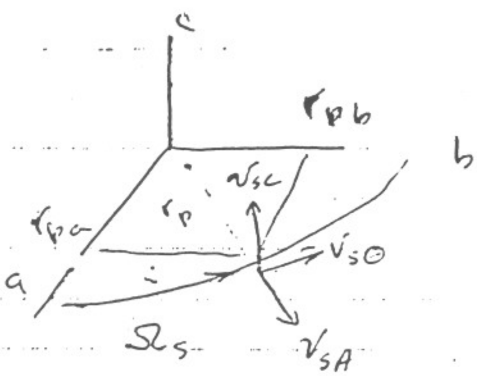


Space ship system -
$\xi \;\eta\; \zeta$

Time of launch T_L

$r_{pa}(T_L)$, $r_{pb}(T_L)$ } position of SS at T_L

$V_{sr}(T_L)$, $V_{s\theta}(T_L)$, $V_{s\zeta}(T_L)$ } polar coordinates of the velocity of the space ship at T_L



$$r_p = \sqrt{r_{pa}^2 + r_{pb}^2}$$

Neglecting the gravitational effect of the launch planet's
Initial condition are for some time after escape.

a_s: $\quad \underline{a_s} = \dfrac{1}{\dfrac{2}{r_p} - \dfrac{1}{\mu}\left(v_{sr}^2 + V_{s\Theta}^2 + V_{sc}^2\right)}$

$$\underline{f_s} = \frac{r_p^2}{\mu}\left(v_{s\Theta}^2 + v_{sc}^2\right)$$

s = space ship.

e_s: $\quad \underline{e_s} = \sqrt{1 - \dfrac{f_s}{a_s}}$

$\operatorname{sgn}(V_{sc})$

Ω_s: $\quad \underline{\Omega_s} \qquad \cos \Omega_s = \dfrac{r_{pa}}{r_p} \qquad \sin \Omega_s = \operatorname{sgn}(V_{sc}) \dfrac{r_{pb}}{r_p}$

if it is the ascending node. (otherwise $\operatorname{sgn}(x)$)

$$\operatorname{sgn}(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

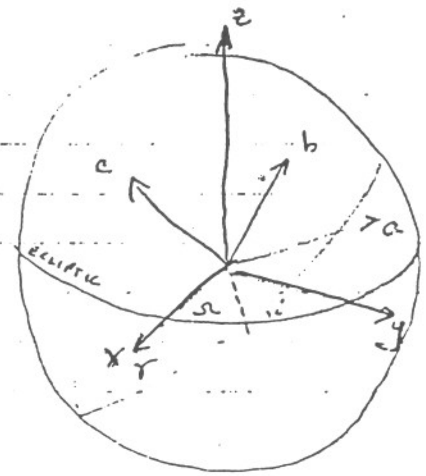
i_s: $\quad \underline{i_s} \qquad \cos i_s' = \dfrac{V_{s\Theta}}{\sqrt{V_{s\Theta}^2 + V_{sc}^2}} \qquad \sin i_s = \dfrac{|V_{sc}|}{\sqrt{V_{s\Theta}^2 + V_{sc}^2}}$

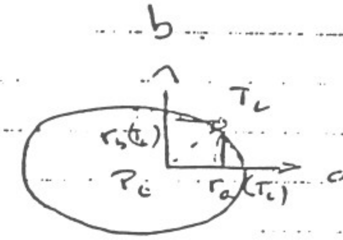
will be pos. because of V_Θ of launching planet.

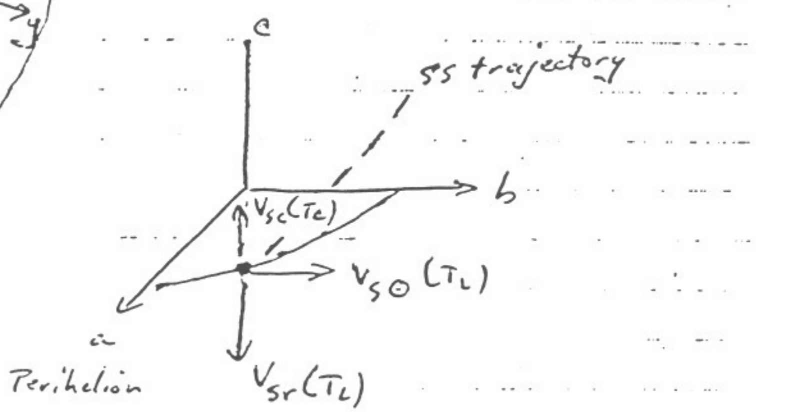
These velocities are initial velocities

ω_s — next time

24 Feb '60 Elements of a Space Ship Trajectory







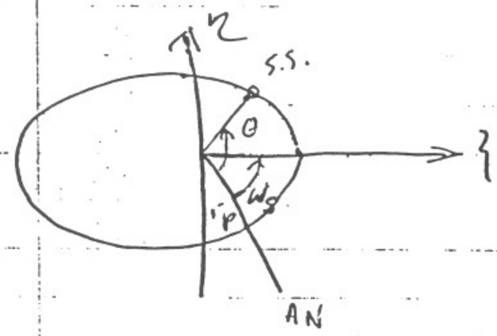
$$r_p = \sqrt{r_{pa}^2 + r_{pb}^2}$$

Space Ship is assumed to have escaped the gravity of its launching planet.

Velocities are inertial velocities with components in polar coord.

Plane of a & b is the plane of the orbit of the launching planet.

Now take ξ, η, ζ so that ξ, η define plane of the space ship orbit with ξ toward perihelion of the space ship.



ω_s :

$$\cos \omega_s = \mathrm{sgn}(V_{sc}) \frac{p_s - r_p}{r_p e_s}$$

$$\sin \omega_s = - \mathrm{sgn}(V_{sc}) \frac{V_{sr} \sqrt{p_s}}{e_s \sqrt{\mu}}$$

$$\begin{bmatrix} r_{sa} \\ r_{sb} \\ r_{sc} \end{bmatrix} = \underset{ss \to \text{planet}}{M_{PS}} \begin{bmatrix} r_{s\xi} \\ r_{s\eta} \\ 0 \end{bmatrix} \qquad M_{PS} = \begin{bmatrix} l_{s1} & l_{s2} & l_{s3} \\ m_{s1} & m_{s2} & m_{s3} \\ n_{s1} & n_{s2} & n_{s3} \end{bmatrix}$$

$l_{s1} = \cos\Omega_s \cos\omega_s - \sin\Omega_s \sin\omega_s \cos i_s$

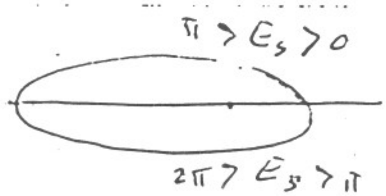
etc.

$$\begin{bmatrix} r_{sx} \\ r_{sy} \\ r_{sz} \end{bmatrix} = \underset{\text{planet} \to \text{ecliptic}}{M_{EP}} \begin{bmatrix} r_{sa} \\ r_{sb} \\ r_{sc} \end{bmatrix}$$

$l_{s1} = f$ [Euler ∡'s for Planet's orbital plane].

etc.

$$\begin{bmatrix} r_{sx} \\ r_{sy} \\ r_{sz} \end{bmatrix} = M_{EP} \cdot M_{PS} \begin{bmatrix} r_{s\xi} \\ r_{s\eta} \\ 0 \end{bmatrix}$$



$r_{s\xi}(t), \quad r_{s\eta}(t)$

(5)
$$r_{s\xi}(t) = a_s[\cos E_s(t) - e_s]$$
$$r_{s\eta}(t) = \sqrt{a_s p_s}\, \sin E_s(t)$$

$$E_s(t) - e \sin E_s(t) = M_s(t) \qquad \text{Keplers Eq.}$$

$$M_s(t) = n_s(t - T_L) + M_s(T_L) \qquad n_s = \sqrt{\frac{a_s^3}{\text{etc}}}$$

(1) $$E_s(T_L) = \pi + \operatorname{sgn}(v_{sr})\left[\cos^{-1}\left(\frac{a_s - r_p}{a_s e_s}\right) - \pi\right]$$

(2) $$E_s(T_L) - e_s \sin E_s(T_L) = M_s(T_L)$$

(3) $$M_s(t) = n_s(t - T_L) + M_s(T_L)$$

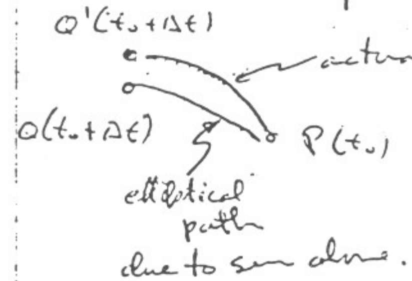
④ $E_s(t) - e_s \sin E_s(t) = M_s(t)$

⑤ $r_{s_\xi}(t) =$

$r_{s_\eta}(t) =$

⑥ Matrix Eq $\rightarrow$ r_{s_x} r_{s_y} r_{s_z}

26 Feb 1960 Effects of distant planets



"oscillating ellipse"

$x = F_1(t_0, a_0, e_0, \Omega_0, \omega_0, i_0, \epsilon_0)$

Expand in a Taylor's Series about x_0

elliptic $\quad x = x_0 + \Delta t\left(\frac{dx}{dt}\right)_0 + \frac{(\Delta t)^2}{2}\left(\frac{d^2x}{dt^2}\right)_0 + \cdots$

actual $\quad x' = x_0' + \Delta t\left(\frac{dx'}{dt}\right)_0 + \frac{(\Delta t)^2}{2}\left(\frac{d^2x'}{dt^2}\right)_0$

$$x' - x = \frac{(\Delta t)^2}{2}\left(\frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2}\right)_0$$

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} = 0$$

$$\boxed{x' - x = \frac{(\Delta t)^2}{2}\left(\frac{\partial R}{\partial x'}\right)_0}$$

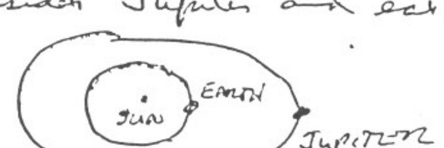
$$\frac{d^2x'}{dt^2} + \frac{\mu x'}{r'^3} = \frac{\partial R}{\partial x'}$$

$$R = Gm_1\left(\frac{1}{\Delta^3} - \frac{x'x_1 + y'y_1 + z'z_1}{r_1'^3}\right)$$

higher terms are neglected.

$$\frac{\partial R}{\partial x'} = Gm_1\left(\frac{x_1 - x'}{\Delta^3} - \frac{x_1'}{r_1^3}\right)$$

Consider Jupiter and earth



$a_{JUP} \cong 5\text{ a.u.}$

$a_{EARTH} \cong 1\text{ a.u.}$

$Gm_1 = Gm_0 \frac{M_1}{} \approx 4\pi^2 \times 10^{-3}$

$\Delta \cong 4$ at worst $\quad r_1 \cong 5 \quad x_1' \cong 5 \quad x_p - x' \cong 4$

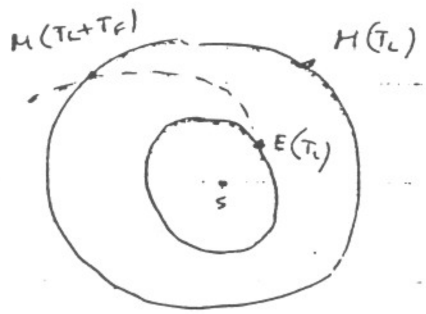
$$x' - x \cong 20 \times 10^{-3}\left(\frac{4}{4^3} - \frac{5}{5^3}\right) \Delta t^2 \cong$$

If $\Delta t = \frac{1}{10}$ years $\quad (x' - x) \sim 4.5 \times 10^{-6}$ a.u.

~ 420 miles

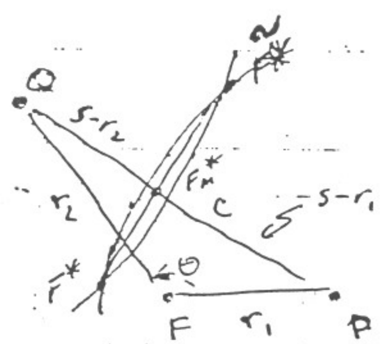
"Osculating Ellipse" is used for short term positional predictions.

Planetary Voyage



Think of T_L & T_F as specified

Then S, E, & M define many ellipses with focus at S & intersecting E & M



Where is the other Focus?

$PF + PF^* = 2a$
$QF + QF^* = 2a$

$PF^* = 2a - r_1$
$QF^* = 2a - r_2$

1) 2 Ellipses with same 2a

What is locus of focus as 2a is changed.

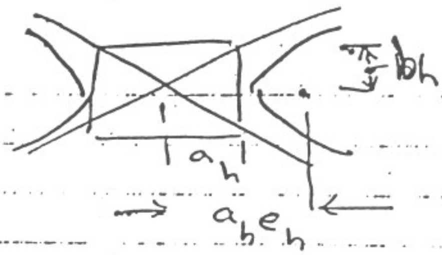
$PF^* - QF^* = r_2 - r_1 = $ const.

F^* lies on a hyperbola with P & Q as foci

Try this with compass & straightedge

a must be > a_m

$$2a_m = \frac{r_1 + r_2 + c}{2} = s$$



$$a_h = \frac{r_2 - r_1}{2} \qquad e_h = \frac{c}{r_2 - r_1}$$

$$\frac{b_h}{a_h} = \pm\sqrt{e_h^2 - 1} = \pm \frac{2\sqrt{r_1 r_2}}{r_2 - r_1} \sin\frac{\Theta}{2}$$

Physical Significance of am

$$v^2 = \mu\left(\frac{2}{r_1} - \frac{1}{a}\right)$$ am defines minimum energy path

This is minimum initial velocity - we may want to minimize velocity we have to give space ship.

Other leaf of hyperbola is locus of vacant focus of hyperbolic paths.

29 Feb 60

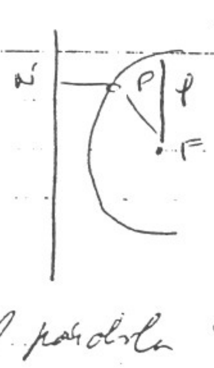
$a > 0$ outside circle of radius r_1
$a < 0$ inside " " "

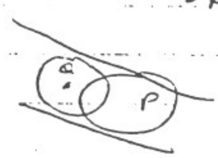
Force would have to be a repelling force so neglect it.

concave
convex

Slope of asymptote - parallel to parabolic paths.

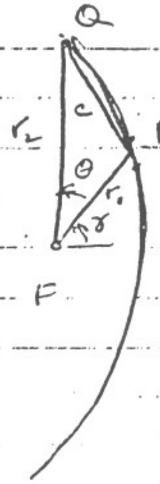
Home Prob. 7





Get Distance by Tangent to circles about P & Q

Equation of parabola $r = \dfrac{p}{1+\cos\Theta}$



$\uparrow \varphi$

$$r = \frac{\ell}{1 + e\cos\varphi}$$

$e = 1$ Parabola
$e < 1$. $\ell = a(1-e^2)$
$e > 1$ $\ell = a(e^2 - 1)$

$$r_1 = \frac{\ell}{1 + e\cos\gamma}$$

$$r_2 = \frac{\ell}{1 + e\cos(\gamma + \theta)}$$

looking for $\ell = \ell(r_1\, r_2\, a\, \Theta)$
$\uparrow$ c effectively

Make change of variables from Lambert's Theorem for time of flight of conic paths. Consider Ellipse alone.

Let $\sin\frac{\alpha}{2} = +\sqrt{\frac{s}{2a}}$; $\sin\frac{\beta}{2} = +\sqrt{\frac{s-c}{2a}}$ $\leftarrow$ use hyperbolic sines for hyperbolic α & $\beta > 0$

$$0 < \beta < \alpha < \pi$$

✓ $c = 2a \sin\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)$

$$\ell = \frac{4a(s-r_1)(s-r_2)}{c^2} \sin^2\left(\frac{\alpha \pm \beta}{2}\right) \quad \text{FOR ELLIPSE}$$

$\uparrow \sinh^2$ for HYPERBOLA

2 March 1960

$+$ sign $\sim F^*$ larger ℓ smaller e
$-$ sign $\sim F^*$ smaller ℓ greater e

$$a \sin^2\left(\frac{\alpha \pm \beta}{2}\right) = \frac{s}{2\cancel{a}}\left(1 - \frac{s-c}{2a}\right) \pm 2\sqrt{\frac{s}{2\cancel{a}}}\sqrt{\frac{s-c}{2\cancel{a}}}\sqrt{1-\frac{s}{2\cancel{a}}}\sqrt{1-\frac{s-c}{2\cancel{a}}} + \frac{s-c}{2\cancel{a}}\left(1-\frac{s}{2a}\right)$$

$\mathcal{L}_{a\to\infty}$ Parabola $\quad \ell = \frac{4(s-r_1)(s-r_2)}{c^2}\left(\sqrt{\frac{s}{2}} \pm \sqrt{\frac{s-c}{2}}\right)^2 \quad \leftarrow$ Geometric Harm Prob.

For $a = a_m = \frac{s}{2}$ $\quad \alpha_m = \pi$

check Battington for "S" formulae

$$\sin^2\left(\frac{\pi \pm \beta_m}{2}\right) = \frac{c}{s}$$

$$\boxed{\ell_m = \frac{2(s-r_1)(s-r_2)}{c}} \rightarrow \boxed{\ell_m = \frac{r_1 r_2}{c}(1 - \cos\Theta)}$$

Lambert's Theorem - 1700's

Let T_F = time of flight of elliptic path $P \to Q$ or ① → ②

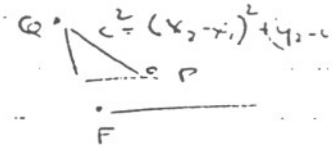
T_F = Function of $a, r_1 + r_2, c$ but not e

$nT = E - e \sin E$ Kepler's Equation

$nT_F = (E_2 - e \sin E_2) - (E_1 - e \sin E_1)$ is the limit of Lambert's Theorem Here

$r_1 + r_2 = a(1 - e \cos E_1) + a(1 - e \cos E_2)$

$c^2 = a^2(\cos E_2 - \cos E_1) + a^2(\sin E_2 - \sin E_1)$



If Lambert's Theorem is true:

$$\left(\frac{\partial T_F}{\partial e}\right)_{a,\, r_1+r_2,\, c} = 0$$

$$nT_F = E_2 - E_1 - 2e \cos\frac{E_2 + E_1}{2} \sin\frac{E_2 - E_1}{2}$$

$$r_1 + r_2 = 2a\left[1 - e\cos\frac{E_2 + E_1}{2} \cos\frac{E_2 - E_1}{2}\right]$$

$$c^2 = 4a^2 \sin^2\frac{E_2 - E_1}{2}\left(1 - e^2 \cos^2\frac{E_2 + E_1}{2}\right)$$

Let $\mu = \dfrac{E_2 + E_1}{2}$ $\quad v = \dfrac{E_2 - E_1}{2}$

Then $nT_F = 2(v - e \cos\mu \sin v)$

$r_1 + r_2 = 2a(1 - e \cos\mu \cos v)$

$c^2 = 4a^2 \sin^2 v (1 - e^2 \cos^2 \mu)$

$\frac{\partial}{\partial e}$, $\frac{\partial}{\partial e}$ solve for $\frac{\partial \mu}{\partial e}$ & $\frac{\partial v}{\partial e}$ & substitute in $\frac{\partial (nT_F)}{\partial e}$

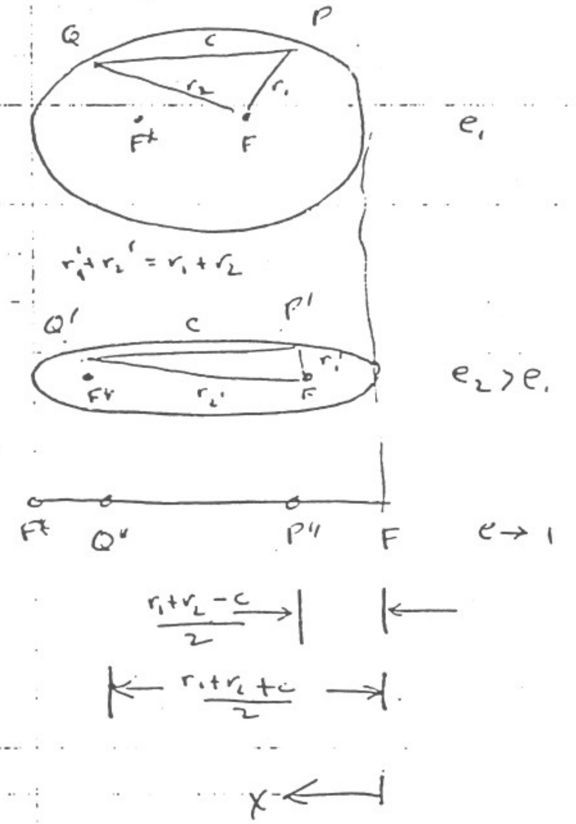
$$\frac{\partial v}{\partial e} = 0 \qquad \frac{\partial \mu}{\partial e} = \frac{\cos\mu}{e \sin\mu}$$

$$\left(\frac{\partial T_F}{\partial e}\right)_{a,\, r_1+r_2,\, c} = 0$$

7 March 1960

Lambert $T_F = T_F(r_1+r_2, c, a)$ independent of e

Useless Alternative {Consideration of this as holding F^* P & Q fixed. T_F is constant as F moves on an ellipse with P & Q as foci



$$P''F + Q''F = r_1 + r_2$$

$$Q''F - P''F = c$$

time of flight from P to Q, $P' \to Q'$, $P'' \to Q''$ all the same.

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

For last case $\left(\frac{dx}{dt}\right)^2 = \mu\left(\frac{2}{x} - \frac{1}{a}\right)$

$$dt = \frac{1}{\sqrt{\mu}} \frac{x\,dx}{\sqrt{2x - \frac{x^2}{a}}}$$

$$T_F = \frac{1}{\sqrt{\mu}} \int_{s-c}^{s} \frac{x\,dx}{\sqrt{2x - \frac{x^2}{a}}}$$

is integrable by tables

or let $x = a(1-\cos\varphi)$

$dx = a \sin\varphi \, d\varphi$

$2x - \frac{x^2}{a} = 2a(1-\cos\varphi) - a(1 - 2\cos\varphi + \cos^2\varphi)$

$= a \sin^2\varphi$

$$\alpha = 2 \sin^{-1}\sqrt{\frac{s}{2a}}$$

$$= \frac{1}{\sqrt{\mu}} \int \frac{a(1-\cos\varphi)\, a \sin\varphi \, d\varphi}{\sqrt{a}\,\sin\varphi}$$

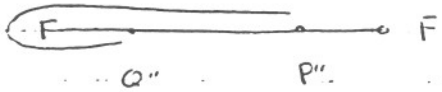
$$\beta = 2 \sin^{-1}\sqrt{\frac{s-c}{2a}}$$

$$T_F = \frac{1}{\sqrt{\mu}}\int_{\alpha}^{\beta} \sqrt{a^3}\,(1-\cos\psi)\,d\psi =$$

$$\boxed{T_F = \sqrt{\frac{a^3}{\mu}}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right]}$$

where $\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}}$ $\quad \sin\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}$

For other case



Criteria is whether the area bounded by the path and the chord of the arc contains the vacant focus.

Show it by fact that if the chord passes through F in original ellipse it remains there as the ellipse is shrunk to a line holding 2a = const.

For Minimum Energy Ellipse
Q'' lies on F

Note all along we have assumed $\angle PFQ < \pi$ and going in counterclockwise direction

$$\tilde{T}_F = T_F + 2\sqrt{\frac{a^3}{\mu}}\int_{\psi=\alpha}^{\psi=\pi} (1-\cos\psi)\,d\psi$$

$$\boxed{\tilde{T}_F = \sqrt{\frac{a^3}{\mu}}\left[2\pi - (\alpha - \sin\alpha) - (\beta - \sin\beta)\right]}$$

9 March 1960

$$\sin\frac{\alpha}{2} = \sqrt{\frac{s}{2a}} \qquad \sin\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}$$

$$\left.\begin{aligned} T_F &= \frac{P}{2\pi}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right] \\ \tilde{T}_F &= \frac{P}{2\pi}\left[2\pi - (\alpha - \sin\alpha) - (\beta - \sin\beta)\right] \end{aligned}\right\} \quad \Theta \leq \pi$$

For all values of Θ

$$T_F = \frac{P}{2}\left\{2N + 1 \cdot \frac{\operatorname{sgn}(\sin\Theta)}{\pi}\left[\pm(\alpha - \sin\alpha - \pi) - (\beta - \sin\beta)\right]\right.$$

N = no of laps around ellipse
$N = 0$ direct path
$N = 1$ one complete circuit + direct path.
$+ \sim F^*$
$- \sim \tilde{F}^*$

For Hyperbolic Paths.

$$T_F = \sqrt{\frac{a^3}{\mu}}\left[(\sinh\alpha - \alpha) - (\sinh\beta - \beta)\right] \qquad \sinh\frac{\alpha}{2} = \sqrt{\frac{s}{2a}}$$

$$\tilde{T}_F = \sqrt{\frac{a^3}{\mu}}\left[(\sinh\alpha - \alpha) + (\sinh\beta - \beta)\right] \qquad \sinh\frac{\beta}{2} = \sqrt{\frac{s-c}{2a}}$$

For Parabolic Paths pass to limit of either ellipse or hyperbola.

$$T_F = \sqrt{\frac{a^3}{\mu}}\left[\alpha - \left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \cdots\right) - \beta + \left(\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} \cdots\right)\right]$$

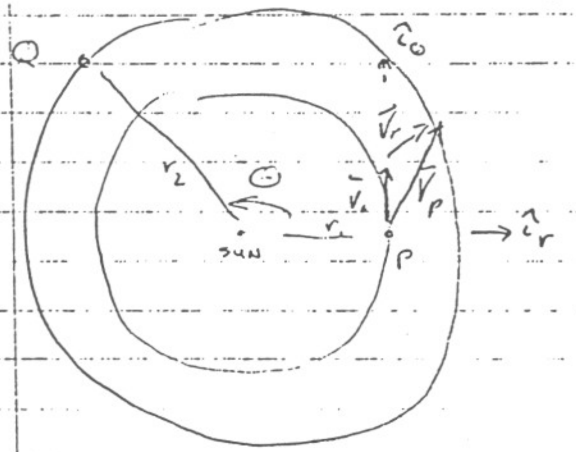
$\frac{\alpha}{2} \sim \sqrt{\frac{s}{2a}}$ $\qquad$ $\frac{\beta}{2} \sim \sqrt{\frac{s-c}{2a}}$ $\qquad$ for a large

$$\underset{a\to\infty}{T_F} = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{3/2} - (s-c)^{3/2}\right]$$

$$\tilde{T}_{\mathrm{I}} = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{3/2} + (s-c)^{3/2}\right]$$ ← EULER (FIRST FOR PARABOLIC COMETS)

$a \to \infty$

__Trajectories in Plane of the Ecliptic__



$v_c^2 = \frac{\mu}{r_1}$ $\quad \vec{V}_R = \vec{V}_P - \vec{V}_c$

$$\vec{V}_R = V_{pr}\hat{c}_r + (V_{p\theta} - V_c)\,\hat{c}_\theta$$

$$V_R^2 = V_{pr}^2 + (V_{p\theta} - V_c)^2$$

Differentiating $r = \frac{\ell}{1+e\cos\theta}$ $\quad V_\theta = \sqrt{\frac{\mu\ell}{r^2}}$

$$v_r^2 + v_\theta^2 = v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

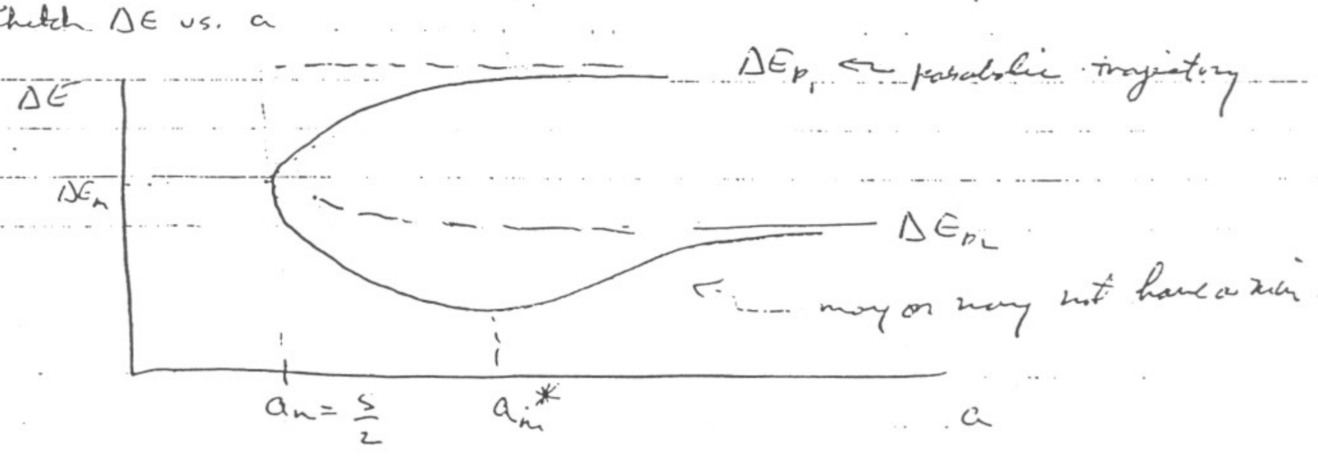
$$V_R^2 = V_c^2\left(3 - \frac{r_1}{a} - 2\sqrt{\frac{\ell}{r_1}}\right) \triangleq V_c^2\,\Delta E$$ ΔE = dimensionless

$$\Delta E = 3 - \frac{r_1}{a} - \frac{\sqrt{a r_2}}{c}\sin\frac{\theta}{2}\sin\left(\frac{\alpha\pm\beta}{2}\right)$$

& using $\ell = \underbrace{\frac{4a(s-r_1)(s-r_2)}{c}}_{\sin\frac{\theta}{2}}\sin^2\left(\frac{\alpha\pm\beta}{2}\right)$

$$\Delta E = 3 - \frac{r_1}{a} - \frac{r_2\sqrt{2r_1}}{c}\sin\theta\left\{\sqrt{\frac{1}{s-c}-\frac{1}{2a}} \pm \sqrt{\frac{1}{s}-\frac{1}{2a}}\right\}$$

Sketch ΔE vs. a



$$\frac{d\,\Delta E}{da} = \frac{r_1}{a^2}\left[1 - \frac{r_2 \sin\theta}{2c\sqrt{2r_1}}\left(\frac{1}{\sqrt{\frac{1}{s-c} - \frac{1}{2a}}} \pm \frac{1}{\sqrt{\frac{1}{s} - \frac{1}{2a}}}\right)\right]$$

There is a ~~[illegible]~~ minimum for ΔE for all inner planets + mass

want to make $\frac{d\Delta E}{da} = 0$ as a criterion for a minimum.

11 March 60

$$\frac{V_R^2}{V_0^2} \equiv \Delta E = 3 - \frac{r_1}{a} - \frac{r_2\sqrt{2r_1}}{c}\sin\theta\left(\sqrt{\frac{1}{s-c} - \frac{1}{2a}} \pm \sqrt{\frac{1}{s} - \frac{1}{2a}}\right)$$

$\frac{d\Delta E}{da}$ above

For large a is $\frac{1}{\sqrt{\frac{1}{s-c} - \frac{1}{2a}}} + \frac{1}{\sqrt{\frac{1}{s} - \frac{1}{2a}}} \overset{?}{<} \frac{2c\sqrt{2r_1}}{r_2 \sin\theta}$.

Proof:

For large a: LHS can be made arbitrarily close to $\sqrt{s-c} + \sqrt{s}$

$$\text{Rhs} > 2\sqrt{2}\,r_1$$

Squaring Both sides $\quad s-c+2\sqrt{s(s-c)} + s \overset{?}{<} 8r_1$

$$\cos\frac{\theta}{2} = \sqrt{\frac{s(s-c)}{r_1 r_2}}$$

$$(\sqrt{r_1}+\sqrt{r_2})^2 \overset{?}{\geq} r_1 + r_2 + 2\sqrt{r_1 r_2}\cos\frac{\theta}{2} \leq 8r_1$$

<u>Sufficient Condition for minimum</u> $\quad \sqrt{r_1} + \sqrt{r_2} \leq 2\sqrt{2r_1}$

$$\sqrt{\frac{r_2}{r_1}} \leq 2\sqrt{2} - 1$$

$\sim 90°$

In case of Jupiter there are values of θ, where minimum energy path is an escape path

Techniques for computing a_m^*

$$\frac{1}{\sqrt{\frac{1}{s-c}-\frac{1}{2a}}}+\frac{1}{\sqrt{\frac{1}{s}-\frac{1}{2a}}}=\frac{2c\sqrt{2r_1}}{r_2 \sin\Theta}$$

$$\frac{1}{\sqrt{\frac{1}{s}-\frac{1}{2a}+\frac{c}{s(s-c)}}}+\frac{1}{\sqrt{\frac{1}{s}-\frac{1}{2a}}}=\frac{2c\sqrt{2r_1}}{r_2 \sin\Theta},$$

Define:

$$\tan\zeta = \frac{\sqrt{\frac{1}{s}-\frac{1}{2a}}}{\sqrt{\frac{c}{s(s-c)}}} \qquad 0 \le \zeta \le \frac{\pi}{2}$$

$$\cos\zeta + \cot\zeta = \frac{2c}{r_2 \sin\Theta}\sqrt{\frac{2cr_1}{s(s-c)}} = \frac{4\left(\frac{c}{r_2}\right)^{3/2}}{\sin\Theta\sqrt{1+\cos\Theta}}$$

Compute this & go through trig tables until $\cos\zeta$ & $\cot\zeta$ add up to the Right Hand Side.

Then

$$2a_m^* = \frac{r_1 r_2 (1+\cos\Theta)}{r_1+r_2-c(\sec^2 z+\tan^2 z)}$$ if this is negative there is no min.

Hohmann ellipse only has condition right very seldom.
& it is not the best in the 3 dimensional case.

14 March 1960

Hohmann Trajectory

Elements of trajectory

$$a_H = \frac{r_1+r_2}{2}$$

$$e_H = \frac{r_2-r_1}{r_2+r_1}$$

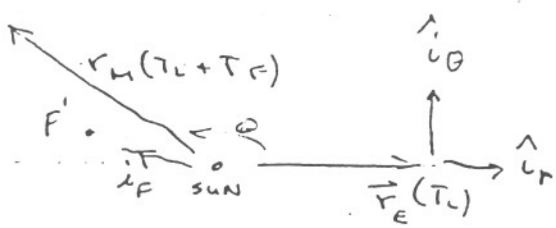
$$\Delta E_H \equiv \frac{V_R^2}{V_c^2} = 3 - \frac{2r_1}{r_1+r_2} - 2\sqrt{\frac{2r_2}{r_1+r_2}}$$

P_1, Q_1, F must be on ~~are not~~ on a line for a free fall orbit

Rare astronomical event for $\Theta = 180°$ & $P_1 Q$ & F in a line.

Assume a time of departure (T_L) & a time of flight (T_F)

$T_F = T_F(a)$



$$T = \frac{P}{2\pi}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right]$$

$$\tilde{T} = P - \frac{P}{2\pi}\left[(\alpha - \sin\alpha) + (\beta - \sin\beta)\right]$$

Let $\pi - \alpha = \lambda$ if $\tilde{T}$ $\qquad 0 \le \lambda \le \pi$

$\alpha - \pi = \lambda$ if T $\qquad -\pi \le \lambda \le 0$

So that Newton iteration for a will always converge.

$$\boxed{T = \frac{P}{2\pi}\left[\pi + \lambda + \sin\lambda - (\beta - \sin\beta)\right] \qquad -\pi \le \lambda \le \pi}$$

$$* \quad \boxed{a = \frac{s}{1+\cos\lambda}}$$

Take T_F to get λ by iteration then use * for a

$$\cos\beta = 1 - \left(\frac{s-a}{s}\right)^2 (1+\cos\lambda) \qquad P = \sqrt{\frac{a^3}{\mu}}$$

$$\frac{dT}{d\lambda} = \frac{3P}{2}\,\frac{\sin\lambda}{1+\cos\lambda} + \frac{P}{2\pi}(1+\cos\lambda)\left[1 + \left(\frac{s-c}{s}\right)^2 \frac{\sin\lambda}{\sin\beta}\right]$$

Newton Iteration Formula

$$\lambda_{n+1} = \lambda_n - \frac{T(\lambda_n) - T_F}{\left(\frac{dT}{d\lambda}\right)_{\lambda=\lambda_n}} \qquad \pi > \lambda_0 > \lambda_F$$

Converges without troubles now

$$p = 4a\,\frac{(s-r_1)(s-r_2)}{c^2}\cos^2\left(\frac{\lambda+\beta}{2}\right)$$

$$\vec{V_S}(T_L) = \frac{dr}{dt}\,\hat{i}_r + r\frac{d\theta}{dt}\,\hat{i}_\theta$$

$$\left(\frac{dr}{dt}\right)^2 = \mu\left(\frac{2r-p}{r^2} - \frac{1}{a}\right) \qquad \left(\frac{r\,d\theta}{dt}\right)^2 = \frac{\mu p}{r^2}$$

Question of sign for $\frac{dr}{dt}$ at launch $\quad \frac{dr}{dt} > 0$ after perihelion, < 0 before perihelion

16 March 1960

$$\left(\frac{dr}{dt}\right)^2 = V_{sr}^2 = \mu\left(\frac{2r_E - p_s}{r_E^2} - \frac{1}{a_s}\right) \quad \text{at } T_L$$

$$\left(r\frac{d\theta}{dt}\right)^2 = V_{s\theta}^2 = \frac{\mu p_s}{r_E^2} \quad \text{at } T_L$$

$\vec{i}_n \perp$ space ship trajectory

$$\mathrm{sgn}\left[(\vec{r}_E \times \vec{i}_F)\cdot \vec{i}_n\right] = \mathrm{sgn}\left(\frac{dr}{dt}\right)$$

$$\vec{i}_F = F_E\,\vec{r}_E + F_M\,\vec{r}_M \qquad \text{Dot with } \vec{r}_E \;\&\; \vec{r}_M \text{ etc.}$$

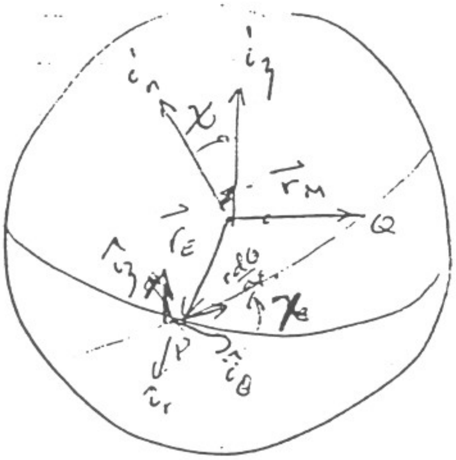
+ MANY STEPS

$$F_E = \frac{1}{e_s r_E \sin^2\theta}\left[\left(1-\frac{\rho}{r_E}\right)-\left(1-\frac{\rho}{r_M}\right)\cos\theta\right]$$

$$F_M = \frac{1}{e_s r_M \sin^2\theta}\left[\left(1-\frac{\rho}{r_M}\right)-\left(1-\frac{\rho}{r_E}\right)\cos\theta\right]$$

$$\cos\theta = \frac{\vec{r}_E \cdot \vec{r}_M}{r_E\, r_M}$$

$$\underline{(\vec{r}_E \times \vec{i}_F)\cdot \vec{i}_M} = \underline{F_M\, r_E\, r_M \sin\theta}$$ to give sign of $\frac{di}{dt}$



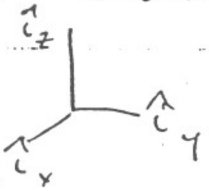
ψ = angle of inclination of s.s. orbit..

Sign of ψ ?

$\psi > 0$ if Q is above the ecliptic and $\theta < 180°$
or if Q is below the ecliptic and $\theta > 180°$
otherwise $\psi < 0$

<u>Summary</u>

Calculate $\vec{r}_E$ & $\vec{r}_M$ in rectangular ~~ecliptic~~ ecliptic coordinates



$$\cos\theta = \frac{\vec{r}_E \cdot \vec{r}_M}{r_E\, r_M}$$

$$\sin\theta = \mathrm{sgn}\left(\vec{r}_E \times \vec{r}_M \cdot \vec{i}_z\right)\sqrt{1-\cos^2\theta}$$

Normal to trajectory plane

$$i_M = \frac{\vec{r}_E \times \vec{r}_M}{r_E\, r_M \sin\theta}$$

$$\cos\chi = \vec{i}_m \cdot \vec{i}_z$$

$$\sin\chi = \operatorname{sgn}(\vec{r}_M \cdot \vec{i}_z \sin\Theta)\sqrt{1-\cos^2\chi}$$

$$\vec{v}_{sp} = \operatorname{sgn}(F_M \sin\Theta)\sqrt{\mu\left(\frac{2r_E - p_s}{r_E^2} - \frac{1}{a_s}\right)}\ \hat{i}_r$$

$$+\sqrt{\frac{\mu p_s}{r_E^2}}\left(\cos\chi\ \hat{i}_\Theta + \sin\chi\ \hat{i}_3\right)$$

<u>Arrival at Mars</u> For 2 dimensional Hohmann Path.

Relative velocity at perihelion : 7460 mph } + Mars gravity
" " aphelion : 4273 mph }

To go in orbit about mars:

V_∞ = vel. along approach hyperbola at a great distance

$$\Delta v = \sqrt{\frac{2\mu_M}{r} + V_\infty} - \sqrt{\mu_M\left(\frac{1+e}{r}\right)}$$

r = distance from mars when we apply Δv
e = eccentricity of orbit about mars

Easier to go into eccentric orbit at mars.

e	Δv at perihelion	Δv at aphelion
0.0	5291 mph	3206 mph
0.2	4840	2755
0.4	4426	2341
0.6	4041	1956
0.8	3679	1594
1.0	3336	1251

18 March 1960 QUIZ NEXT FRIDAY

At the point Q (arrival at Mars)... in planet coordinates

$$\bar{v}_{SQ} = \operatorname{sgn}(-F_{\varepsilon} \sin\Theta) \sqrt{\mu\left(\frac{2r_M - \ell}{r_M^2} - \frac{1}{a}\right)}\, \bar{i}_p$$

$$= \frac{\sqrt{\mu f_s}}{r_M}\left(\cos\nu\, \bar{i}_q + \sin\nu\, \bar{i}_q\right)$$

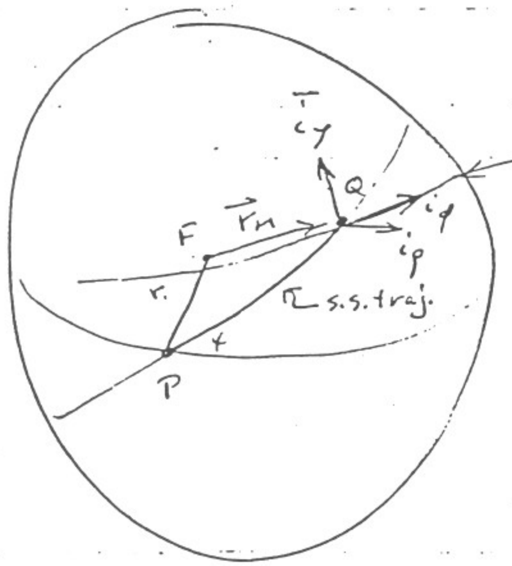
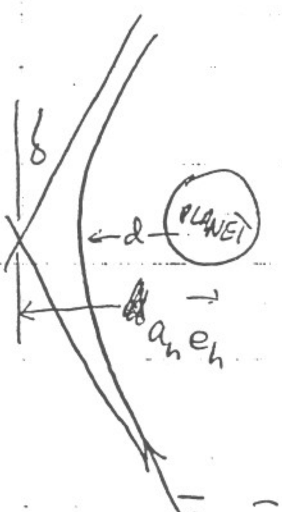


Figure in R-219

$$\cos\nu = i_n \cdot i_y$$

$$\sin\nu = \operatorname{sgn}(-\vec{r}_{\varepsilon} \cdot i_\varphi \sin\Theta)\sqrt{1-\cos^2\nu}$$



$$v^2 = \mu_p\left(\frac{2}{r} + \frac{1}{a}\right) \quad \text{hyperbola}$$

$$v_\infty^2 = \frac{\mu_p}{a_h} \qquad a_h = \frac{\mu_p}{v_\infty^2}$$

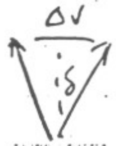
$$e_h = \sqrt{1+\left(\frac{b_h}{a_h}\right)^2} \qquad \frac{b_h}{a_n} = \cot\delta$$

$$e_h = \csc\delta$$

$$\bar{V}_\infty = \bar{V}_{SQ} - \bar{V}_{PLANET}$$

$$\boxed{\sin\delta = \frac{1}{1 + \dfrac{v_\infty^2 d}{\mu_p}}}$$

$\boxed{\Delta V = 2 V_\infty \sin\delta}$ ⟵ rotation of the relative velocity vector
note that the inertial velocity
is changed in magnitude.

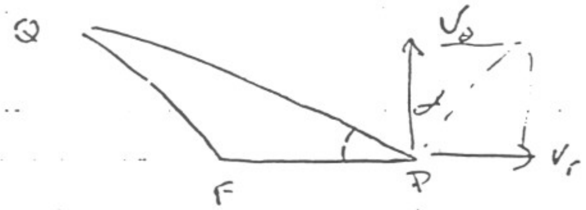


21 March 1960

Problems.

#1 Let V_r & V_θ be the polar coordinate components of velocity of the vehicle at point P.
Show that for the minimum energy path:

$$\frac{V_r}{V_\theta} = \tan\left(\tfrac{1}{2} \angle FPQ\right)$$



$\alpha = \frac{1}{2} \angle QPF$

#2 P 37 Eq 2.2-8 factor of r_2 in denominator

$$\frac{\quad}{r_2(1+\cos\theta)}$$

#3 Trajectory calculation in R-219 is wrong
don't use slide rule

<u>Round trip trajectory calculation</u>

Given T_F there is one a if you go on a <u>direct elliptical</u>
two values if you take 2 laps to get there.
⟵ one way trip

Round trip

1. Assume a value for T_L , V_{RE} ← initial cond.

2. Guess at value of T_F ← fixes the [illegible] & orbit

3. Calculate a from T_F

4. Calculate e from a

5. Determine $\bar{V}_{SP}$, V_{EP} , $V_{RE} = V_{SP} - V_{EP}$

6. Compare computed V_{RE} with assumed V_{RE} Do not match V_{RE}

7. If different change T_F & repeat

8. Calculate $\bar{V}_{SQ}$ $\bar{V}_{MQ}$ & $\bar{V}_{RM} = \bar{V}_{SQ} - \bar{V}_{MQ}$

relative to Mars. $\bar{V}_{RM}$

9. Repeat 2-7 with

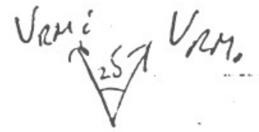
$T_L + T_F$ as time of launch from Mars

$|\bar{V}_{RM}|$ as the departure velocity magnitude

mars only rotates the relative velocity vector

10. Inbound Velocity Vector V_{RM_i}

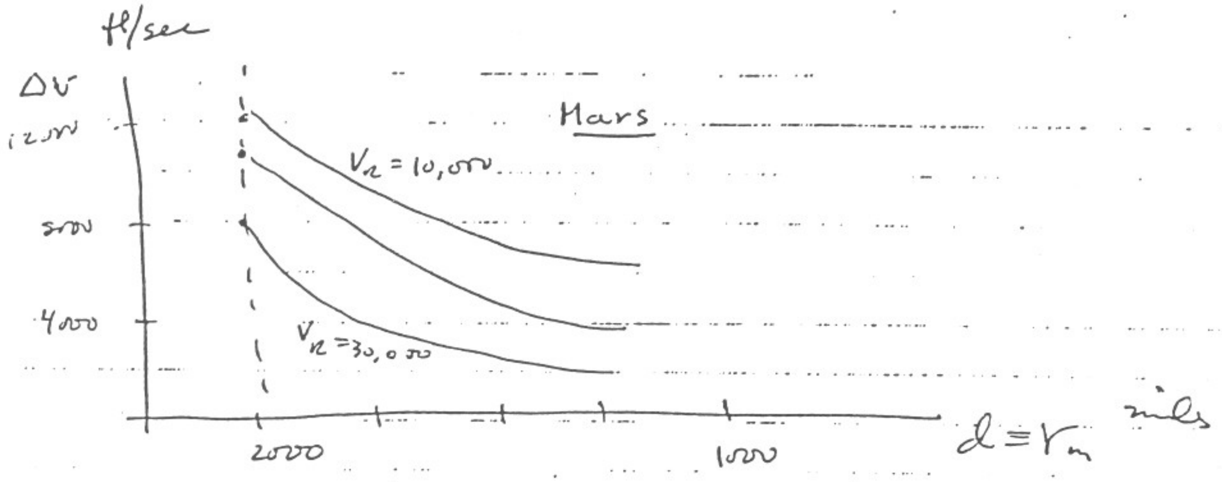
Outbound Velocity Vector V_{RM_o}



$$\sin 2\delta = \frac{|V_{RM_o} \times V_{RM_i}|}{V_{RM}^2}$$

$$d = \frac{\mu_M}{V_{RM}^2}(\csc\delta - 1)$$

No guarantee that you can get a solution by this technique however.

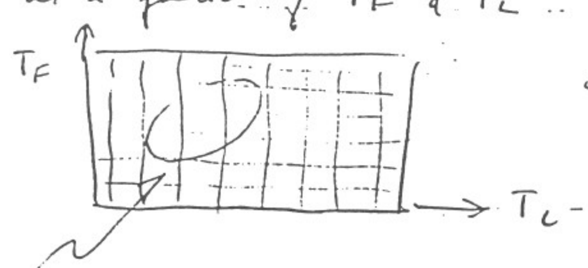


Possible to go to both Venus & Mars in 1.8 years in 1965 or so. with about 16,000 ft/sec & return to earth

Quiz Friday   CLOSED BOOK   Be able to describe the procedure for a calculation.

23 March '60

Generate all possible paths to Mars & Venus as a function of T_F & T_L



Doing this now on 704

Contours of Constant Launch Velocity or Arrival Velocity
Can then pick a time when conditions are most favorable

Still have to worry about where you fire it
(injection from over Siberia might be prohibitive.

6 year Periodicity of Earth, Venus & Mars.

like Beat Frequency

Synodic Period   S   E M

Time to repeat relative position   2.15 years. EM
1.7 years E Venus
~6 years for EVM

Ehricke - idea from new book (original with

≤ ~~[illegible]~~ activity sphere - outside you consider the planet as disturbance, etc. in mile consider the sun as disturbance.

$$r_{ac} = r_p \left(\frac{\text{mass of planet}}{\text{mass sun}} \right)^{2/5}$$

in AU. r_{ac} = .00618 Earth
= .00412 Venus
= .00378 Mars.
7.5 for Neptune

4 April 1960

4 weeks more of Dr. Battin's Astro Nav. - then Dr. Wrigley - Relativity

Consider problem of self contained navigation - getting a fix by a star or planet shot

Need 3 angles to get fix - each one puts you on a cone.

Indeterminancy of a.u. is of same order of magnitude as what inaccuracy you will have in your measurement

Have a program of what angles should be at every time & then measure an error from what you should have

Have an error box "box of uncertainty" just like Δ in conventional navigation.

T = time at which the fix should be made
δT = error in clock
$T - \delta T$ = time at which fix is ~~made~~ started
δt = time to make the fix $\quad T - \delta T + \delta t$ = time angle is measured

I Angle from the sun to a planet

Prob:- determine δA = Deviation in measured $\angle$ due to:
1) Motion of planet during $\delta t - \delta T$
2) Initial Displacement $\delta \underline{r}$ at time $T - \delta T$
3) Distance travelled by space ship with velocity $\underline{V}_s$ during the time $(\delta t - \delta T)$

$\underline{r}$: vector from Sun to S_0
$\underline{z}$: vector from S_0 to P_0

$$\cos A = - \underline{r} \cdot \underline{z} / r z$$

$$\delta(r z \cos A) = -\delta(\underline{r} \cdot \underline{z}) = -\underline{r} \cdot \delta \underline{z} - \underline{z} \cdot \delta \underline{r}'$$

$$\delta \underline{r}' = \delta \underline{r} + \underline{v}_s \delta t$$

$$\delta \underline{z} = -\underline{V}_p (\delta T - \delta t) - \delta \underline{r}'$$

vector underneath because superscripts are coming later

$$\delta(r z \cos A) = -r z \sin A \delta A + \cos A (r \delta z + z \delta r)$$

$$\delta r = \frac{\underline{r} \cdot \delta \underline{r}'}{r} \qquad \delta z = \frac{\underline{z} \cdot \delta \underline{z}}{z}$$

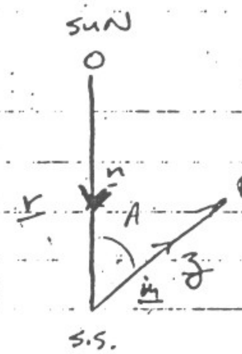
Now substitute to get δA in terms of $\delta r'$ & δz

$\therefore$ in terms of $\delta \underline{r}, \delta T, \delta t$

$\delta \underline{r}$ = deviation of the SS from reference point at $T - \delta T$

$$\delta r' = \delta \underline{r} + \underline{v}_s \delta t \qquad \delta z = \underline{V}_p (\delta t - \delta T) - \delta \underline{r}'$$

6 April 1960



$\underline{m}$: SS to P_0

$\underline{n}$: SS to Sun

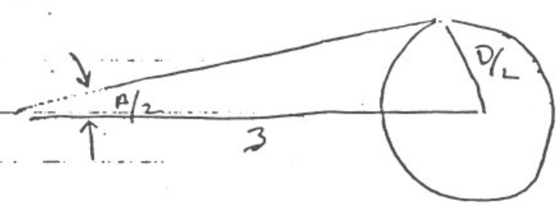
$dr = -dz$

$$\delta A = \frac{\underline{m} - (\underline{n}\cdot\underline{m})\,\underline{n}}{r \sin A} \cdot \delta \underline{r} - \frac{\underline{n} - (\underline{n}\cdot\underline{m})\,\underline{m}}{z \sin A} \cdot \delta \underline{z}$$

$$\delta A = f(\delta \underline{r}\, ; \delta T, \delta t)$$

where δt is known ahead & we may read that in first. (δt isn't somewhat coef. in the ephemeris are known for each fit)

II Angular Diameter Measurement



$$\sin \frac{A}{2} = \frac{D}{2z}$$

$$\delta A = -\frac{D}{z^2 \cos \frac{A}{2}}\ \underline{m} \cdot \delta \underline{z}$$

Weighting of the measurements

m = no. of measurements $m > 4$ where measuring time is one of the measurements.

$$\delta \underline{A}_m = \begin{pmatrix} \delta A_1 \\ \vdots \\ \delta T \\ \delta A_5 \\ \vdots \\ \delta A_n \end{pmatrix}$$

m = dimension of vector

where one of δA's is δT

$$\delta \underline{r}_4 = \begin{pmatrix} \delta \underline{r} \\ \delta T \end{pmatrix} \qquad \boxed{\delta \underline{A}_m = U_{m4}\, \delta \underline{r}_4}$$

subscripts are the dimensions of the matrix

$$U_{m4} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & — & & \\ u_{31} & — & & \\ 0 & 0 & 0 & 1 \\ u_{51} & — & — & \\ \vdots & & — & \end{bmatrix}$$

so that one equation is $\delta T = \delta T$

$$\begin{bmatrix} \delta A_1 \\ \delta A_2 \\ \delta A_3 \\ \delta T \\ \delta A_5 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & — & — & \\ u_{31} & — & — & — \\ 0 & 0 & 0 & 1 \\ | & \backslash & | & \\ | & | & & \end{bmatrix} \begin{bmatrix} \delta r_1 \\ \delta r_2 \\ \delta r_3 \\ \delta T \end{bmatrix}$$

$$\delta \underline{\tilde{A}}_m = \text{measured deviations} = \begin{bmatrix} \delta \tilde{A}_1 \\ \delta \tilde{A}_2 \\ \delta \tilde{A}_3 \\ 0 \\ | \\ | \\ | \end{bmatrix}$$

← measured value.

← Best guess for error in clock is zero.

$$\delta \underline{\tilde{r}}_4 = \begin{bmatrix} \delta \underline{\tilde{r}} \\ \delta \tilde{T} \end{bmatrix} = \text{estimated position deviation \& clock error}$$

$$\delta \underline{\tilde{A}}_m = \delta \underline{A}_m + \underline{\alpha}_m$$ ← error in ability to measure (noise if you like)

$$\delta \underline{\tilde{r}}_4 = \delta \underline{r}_4 + \underline{\epsilon}_4$$ ← uncertainty in our position after we compute the fix

Method of Maximum Likelihood

Assume $\alpha_1, \alpha_2, \ldots \alpha_m$ are jointly normal random variables with zero mean

$$f(\alpha_1 \alpha_2 \alpha_3 \ldots) = \frac{1}{\sqrt{(2\pi)^m \, |\Phi_{mm}|}} \exp\left[-\tfrac{1}{2} \underline{\alpha}_m^T \, \Phi_{mm}^{-1} \, \underline{\alpha}_m\right]$$

Moment Matrix $\Phi_{mm} = \begin{bmatrix} \overline{\alpha_1\alpha_1} & \overline{\alpha_1\alpha_2} & \cdots & \overline{\alpha_1\alpha_m} \\ \overline{\alpha_2\alpha_1} & \ddots & & \\ & & \text{mean values} & \\ & & \ddots & \overline{\alpha_m\alpha_m} \end{bmatrix}$

"Correlation Matrix"

$$\underline{\alpha}_m = \delta \underline{\tilde{A}}_m - U_{m4} \, \delta \underline{r}_4$$

& substitute in Φ_{mm}

then pick elements of $\underline{r}_4$ to maximize the density function

$L(\delta r_1 \; \delta r_2 \; \delta r_3 \; \delta T)$ ← Likelihood Function after we substitute in.

8 April 1960

Matrices - singular matrix implies you are transforming into coordinate system from which you cannot return. (i.e. vector into its projection).

~ - measured quantities
No ~ - actual quantities

In practice $m = 7$ 6 measured angles

Assume $\underline{\alpha}_m$ has a known correlation matrix

$$\Phi_{mm} = \begin{bmatrix} \overline{\alpha_1\alpha_1} & \overline{\alpha_1\alpha_2} & \text{---} & \text{---} \\ | & & & \\ | & & & \overline{\alpha_n\alpha_m} \end{bmatrix}$$

$\underline{\alpha}_m$ is a random vector (elements are random variables

α's are jointly normal

(1) $$f(\alpha_1 \cdots \alpha_m) = \frac{1}{\sqrt{(2\pi)^n |\Phi_{mm}|}} \exp\left(-\frac{1}{2}\underline{\alpha}_n^T \Phi_{mm}^{-1} \underline{\alpha}_m\right)$$

(2) $$\underline{\alpha}_m = \delta\underline{\tilde{A}}_m - U_{m4}\,\delta\underline{r}_4$$

Substitute (2) into (1) to get a quantity

$$L(\delta\tilde{r}_1\,\delta\tilde{r}_2\,\delta\tilde{r}_3\,\delta\tilde{T})$$

Pick $\delta\tilde{r}$'s which maximize L. Then we get the most likely position

Differentiate partially wrt each $\delta\tilde{r}_i$ and set $=0$

$$\underline{\nabla} = \begin{bmatrix} \frac{\partial}{\partial(\delta r_1)} \\ | \\ \frac{\partial}{\partial\,\delta r_i} \end{bmatrix} \qquad \underline{\nabla} L = 0$$

$U_{4m} = U_{m4}^T$

(3) $$U_{4m}\Phi_{m}^{-1}U_{m4}\,\delta\underline{\tilde{r}}_4 = U_{4m}\,\underline{\Phi}_{mm}^{-1}\,\delta\underline{\tilde{A}}_m \qquad 4\times 1 \text{ vector eq.}$$

$\sim$ on r's is best estimate

$\sim$ on A's is measured quantity

Careful that there are 2 $\tilde{T}$'s

measured $\tilde{T}=0$ but best estimate $\Delta\tilde{T}\neq 0$

$\overline{\alpha_i \alpha_j} = 0$ if measurements are independent.

Φ_{mm} = diagonal matrix

$$= \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \sigma_3^2 & \\ 0 & & & \ddots \sigma_m^2 \end{bmatrix}$$

Mean squared estimate (curve fitting)

$$\frac{\left[\delta\tilde{A}_1 - (u_{11}\delta r_1 + \cdots u_{14}\delta T_0)\right]^2}{\sigma_1^2} + \frac{\left[\delta\tilde{A}_2 - (\quad \right]^2}{\sigma_2^2} + \frac{[\quad]^2}{\sigma_m^2} =$$

Chose δr_i to minimize the Σ of squares.

of amounts by which equations are off

with each weighted by

Mean squared analysis & method of maximum likelihood

they are equivalent ~~where~~ 1) variables are independent
2) jointly normally distributed

11 April 1960

(3) $$U_{4m} \Phi^{-1}_{mm} U_{4m} \, \delta \underline{\tilde{r}}_4 = U_{4m} \Phi^{-1}_{mm} \, \delta \underline{\tilde{A}}_m$$

Only need α's independent for mean squared analysis
not necessary for maximum likelihood

$$\delta \underline{\tilde{r}}_4 = \left(U_{4m} \Phi^{-1}_{mm} U_{m4}\right)^{-1} U_{4m} \Phi^{-1}_{mm} \, \delta \tilde{A}_m$$

$$\delta \underline{\tilde{r}}_4 = \begin{bmatrix} \delta \tilde{r}_1 \\ \delta \tilde{r}_2 \\ \delta \tilde{r}_3 \\ \delta \tilde{T} \end{bmatrix}$$

$\delta \tilde{T} > 0$ fix was too early

$\delta T < 0$ fix was too late

Drift rate on clock is 10^{-5} rms hours/hour
after one measurement we have a guess at the clock error

$$\delta\tilde{\underline{r}}_4(T) = \overbrace{\begin{bmatrix} 1 & 0 & 0 & v_{s1} \\ 0 & 1 & 0 & v_{s2} \\ 0 & 0 & 1 & v_{s3} \\ 0 & 0 & 0 & 1 \end{bmatrix}}^{4\times 4} \delta\tilde{\underline{r}}_4$$ ← correction in position for clock-error

Want to
Rewrite Equation (3) — note that a perfect measurement makes Φ singular

Partition Φ_{mm} — where r = no. of redundant measurement
$r = m - 4$

$$\Phi_{mm} = \left\| \begin{matrix} \Phi_{44} & C_{4r} \\ C_{r4} & \Phi_{rr} \end{matrix} \right\| \qquad U_{m4} = \left\| \frac{U_{44}}{U_{r4}} \right\|$$

$$P_{44} \triangleq \left(U_{44}^{-1}\, \Phi_{44}\, U_{44}^{T\,-1}\right)^{-1} \qquad \text{Note: } (A^T)^{-1} = (A^{-1})^T$$

$$Q_{rr} = \Phi_{rr} + U_{r4}\, \cancel{\Phi_{44}}\, P_{44}\, U_{4r}$$

Then:

$$\left(U_{4m}\, \Phi_{mm}\, U_{m4}\right)^{-1} = P_{44} - P_{44} U_{4r} Q_{rr}^{-1} U_{r4} P_{44}$$

$$U_{44} = \left\| \begin{array}{c|c} U_{33} & \begin{matrix} u_{14} \\ u_{24} \\ u_{34} \end{matrix} \\ \hline 0\ 0\ 0 & 1 \end{array} \right\|$$

$$U_{44}^{-1} = \left\| \begin{array}{c|c} U_{33}^{-1} & -U_{33}^{-1} \begin{vmatrix} u_{14} \\ u_{24} \\ u_{34} \end{vmatrix} \\ \hline 0\ 0\ 0 & \cancel{F}\ 1 \end{array} \right\|$$

Θ = zero matrix

(5) $$\boxed{\delta \tilde{\underline{r}}_4 (T) = X_{44}\, U_{44}^{-1} \Big\{ \| I_{44} \vdots O_{4r} \| + B_{4r} Q_{rr}^{-1} \| -A_{r4} \vdots I_{rr} \| \Big\} \delta \tilde{\underline{A}}_m}$$

where:

$$A_{r4} = U_{r4}\, U_{44}^{-1}$$

$$B_{4r} = \Phi_{44} A_{4r} \qquad Q_{rr} = \Phi_{rr} + A_{r4} B_{4r}$$

2nd term in 5 is the redundant measurements
note that 1st 4 measurements are independent of the statistics.

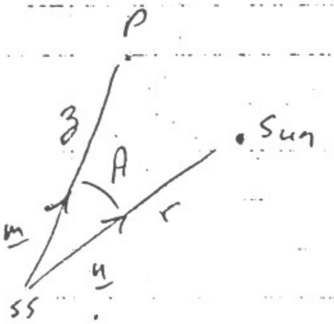
What effect does the geometry have on the accuracy of a fix?

<u>13 April 1960</u>

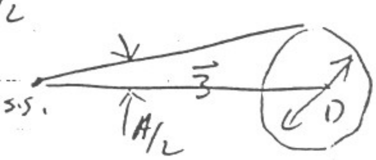
Neglect any errors in the clock & make 3 angular measurements -

r = ∞ for star   ρ = ∞ for star

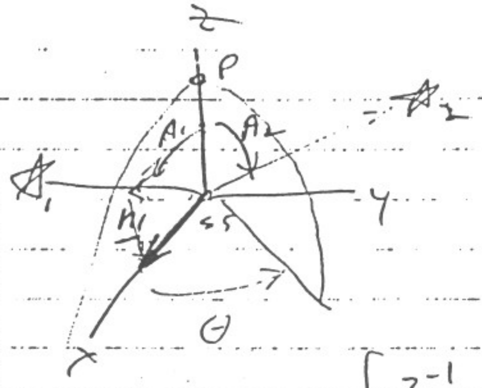
$$\delta A^i = \left(\frac{\underline{m} - (\underline{n} \cdot \underline{m})\, \underline{n}}{\underline{r} \sin A} + \frac{\underline{n} - (\underline{m} \cdot \underline{n})\, \underline{m}}{\rho \sin A} \right) \cdot \delta \underline{r}$$



$$\delta A = \frac{D\, \underline{m} \cdot \delta \underline{r}}{\rho^2 \cos A/2}$$



I
1) Planet & star 1
2) Planet & star 2
3) Angular diameter of planet.



$$\delta A = \bar{n}_1 \cdot \delta \underline{r}$$

$$\bar{n}_1 = \frac{n - (\bar{m} \cdot \bar{n})\bar{m}}{s \sin A}$$

$$U_{33} = \begin{bmatrix} s^{-1} & 0 & 0 \\ s^{-1}\cos\theta & s^{-1}\sin\theta & 0 \\ 0 & 0 & s^{-1} \end{bmatrix} \qquad s = \frac{s^2 \cos \theta/2}{D}$$

$$\delta \tilde{\underline{r}}_4(t) = X_{44}\, U_{44}^{-1} \left\{ \| I_{44} O_{4r} \| + \cdots \right\} \delta \underline{\tilde{A}}_m$$

without Time

$$\delta \underline{r} = U_{33}^{-1}\, \delta A_3 \qquad \widetilde{\delta \underline{A}_m} = \delta \underline{A}_m + \alpha_m$$

$$\delta \tilde{\underline{r}}_4 = \delta \underline{r}_4 + \underline{\epsilon}_4$$

$$\boxed{\underline{\epsilon}_3 = U_{33}^{-1}\ \underline{\alpha}_3}$$

Look at mean squared error in position

$$\cancel{= \overline{\epsilon_3 \cdot \epsilon_3} = \overline{\epsilon_3^2} = \underline{\epsilon}_3^T \underline{\epsilon}_3}$$

$$\overline{\epsilon_3 \epsilon_3^T} = U_{33}^{-1}\ \overline{\underline{\alpha}_3 \underline{\alpha}_3^T}\ U_{33}^{T\,-1}$$

Mean squared error $= \overline{\underline{\epsilon}_3^2} = \operatorname{trace}\left(U_{33}^{-1}\, \Phi_{33}\, U_{33}^{T-1} \right)$

$$\mathcal{U}_{33}^{-1} = \begin{bmatrix} z & 0 & 0 \\ -z\cot\theta & z\csc\theta & 0 \\ 0 & 0 & s \end{bmatrix} \qquad \Phi_{33} = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$$

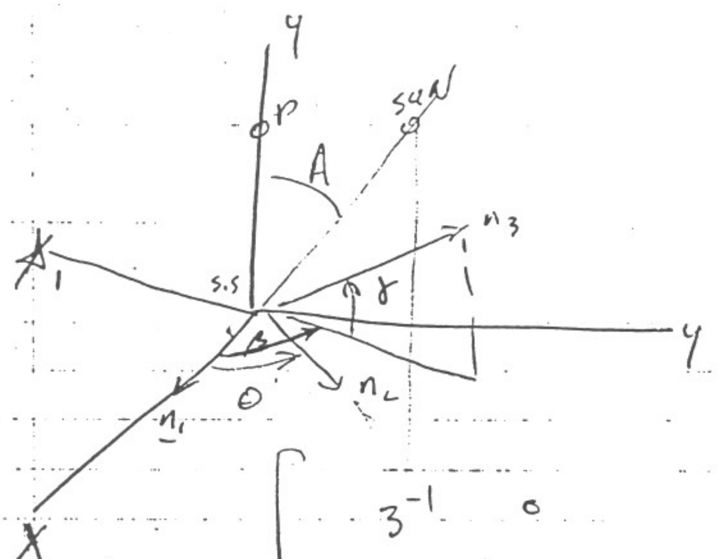
$$\bar{\epsilon}^2 = \sigma_1^2 z^2(1+\cot^2\theta) + \sigma_2^2 z^2 \csc^2\theta + \sigma_3^2 s^2$$

$\bar{\epsilon}^2 = \min$ for $\theta = 90°$ $\qquad s = \frac{z}{2D}\sqrt{4z^2 - D^2}$

$$\boxed{\bar{\epsilon}^2 = (\sigma_1^2 + \sigma_2^2) z^2 + \sigma_3^2 z^2 \left[\frac{z^2}{D^2} - \frac{1}{4}\right]}$$

II ; Planet & Star 1
4 Planet & Star 2
3) Sun & Star 3

$\underline{n}_3$ ⊥ to Sun line & in plane of the meas. 3.



$$\mathcal{U}_{33} = \begin{bmatrix} z^{-1} & 0 & 0 \\ z^{-1}\cos\theta & z^{-1}\sin\theta & 0 \\ r^{-1}\cos\gamma\cos\beta & r^{-1}\cos\gamma\sin\beta & r^{-1}\sin\gamma \end{bmatrix}$$

$$\bar{\epsilon}^2 = \sigma_1^2 z^2\left[1 + \cot^2\theta + \cot^2\gamma \csc^2\theta \sin^2(\beta-\theta)\right]$$
$$+ \sigma_2^2 z^2(\csc^2\theta + \cot^2\gamma \csc^2\theta \sin^2\beta) + \sigma_3^2 r^2 \csc^2\gamma$$

For min $\overline{\epsilon^2}$ make $\gamma_{\text{eff}} = \gamma_{\max} = A$

Assume $\sigma_1 = \sigma_2$ Diff wrt β partially

Optimum $\beta = \frac{1}{2}\theta$

$$\text{Min } \overline{\epsilon^2}(\theta) = \sigma_1^2 z^2 \csc^2\theta \left[2 + \cot^2 A (1-\cos\theta)\right] + \sigma_3^2 r^2 \csc^2 A$$

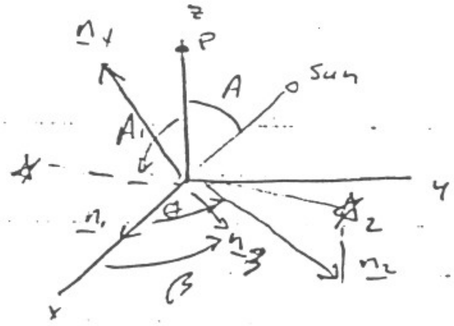
Now θ optimum given by

$$\cos\theta_{opt} = \frac{1-\sin A}{1+\sin A}$$

$$\boxed{\text{Absolute min } \overline{\epsilon^2} = \frac{1}{2}\sigma_1^2 z^2 (1+\sin A)^2(1+\csc A - \sin A) + \sigma_3^2 r^2 \csc^2 A \qquad \sigma_1^2 = \sigma_2^2}$$

15. April 1960

III Planet - Star$_1$
Planet - Star$_2$
Sun - Planet



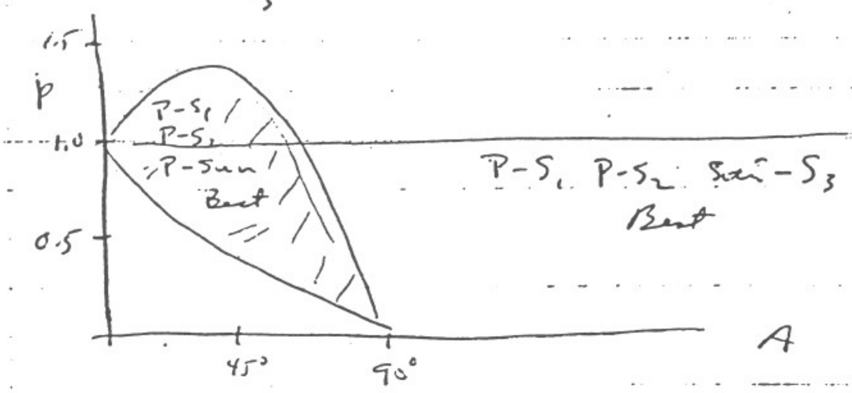
n_3 n_4 P & S all in one plane.

Optimum $\beta = \frac{\theta}{2}$

Opt θ. $\cos\theta_{opt} = \dfrac{\sqrt{1-2p\cos A + p^2} - \sin A}{\sqrt{1-2p\cos A+p^2} + \sin A}$ $\qquad p = \frac{r}{z}$

$\delta A = \left(\frac{n_3}{3} + \frac{n_4}{r}\right)\cdot \delta\bar{r}$ $\Leftarrow$ one row of U_{33}

$$\text{Min } \overline{\epsilon^2} = \sigma_1^2 z^2 \csc^2 A \left(\sin A + \sqrt{1 - 2p\cos A + p^2}\right)^2 / 2$$
$$+ \sigma_3^2 r^2 \csc^2 A$$

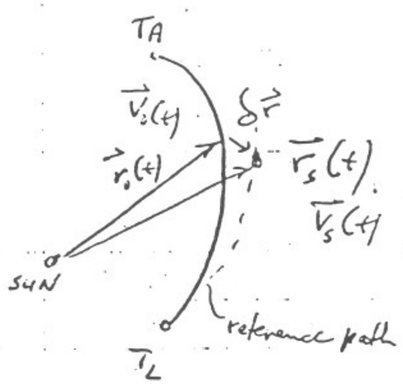


Plot locus of equal $\overline{\epsilon^2}$ for 2 measurements.

Usually above $p=1$ is region of interest. Below $p=1$ you would use all stars, probably

<u>Navigation</u> 1. Get to a point at the right time
2. Just get there ← reduces fuel by factor of 2

1. is important if you want to return & increase ~~miss~~ probability



$$\delta\vec{r}(t) = \vec{r}_s(t) - \vec{r}_0(t)$$
$$\delta\vec{V}(t) = \vec{V}_s(t) - \vec{V}_0(t)$$

$\vec{V}_s^*(t)$ = velocity required at ($\vec{r}_s(t)$ to arrive at target at reference time)

Taylor's Expansion $\vec{V}_s^*(\vec{r}_s(t), t) = \vec{V}_0(\vec{r}_0, t) + C^*(\vec{r}_0, t)\,\delta\vec{r}(t) + \cdots$

$$C^*(r_0, t) = \left[\frac{\partial v_{s i}^*}{\partial r_{s j}}\right] \qquad \vec{V}_s^*(t) = \begin{bmatrix} V_{sx}^*(t) \\ V_{sy}^*(t) \\ V_{sz}^*(t) \end{bmatrix}$$

↑ target point, ↑

$$\vec{V}_s(\vec{r}_s(t)\,t) = \vec{V}_0(\vec{r}_0, t) + C(\vec{r}_0, t)\,\delta\vec{r}(t)$$

$$C(\vec{r}_0\,t) = \left[\frac{\partial v_{si}}{\partial r_{sj}}\right]$$ ← launch pt. is constant

$\bar{\Delta}(t)$ = velocity to be applied at time t

$$\boxed{\bar{\Delta}(t) = (C^* - C)\,\delta\vec{r}(t)}$$

18 April 1960

Fundamental Matrices

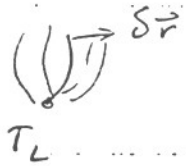
$\underline{\underline{C^*}}$

to be applied to arrive at T_A

$$C^*(t)\;\delta\vec{r} = \delta\vec{v}^*$$

C^* is that matrix that produces the $\delta\vec{v}$ when it acts on $\delta\vec{r}$

$\underline{\underline{C}}$



$$\delta\vec{v} = C(t)\;\delta\vec{r}$$

velocity difference from reference in order that we got displaced $\delta\vec{r}$

$$\boxed{\vec{\Delta}(t) = \left[C^*(t) - C(t)\right]\delta\vec{r}}$$ velocity increment to be added

$\delta\bar{v}(T_L)$ = launch velocity error.

$$\bar{r}_s(\bar{v}_s(T_L)\,;\,t) = \bar{r}_0(\bar{v}_0(T_L)\,;\,t) + R(\bar{v}_0(T_L\,;\,t)\,\delta\bar{v}(T_L)$$

$$\delta\bar{v}(T_L) = \bar{v}_s(T_L) - \bar{v}_0(T_L)$$

$$R = \left\| \frac{\partial r_s^{\,i}}{\partial (\cancel{\xi} v(T_L))^j} \right\|$$

<u>Note:</u> $\overline{\Delta(t)} = \left[C^*(t) - C(t) \right] R(t) \; \delta\bar{v}(T_L)$

Let $\vec{v}_s(T_A)$ = actual velocity after assumed

$$\boxed{\bar{r}_s(\bar{v}_s(T_A); t) = \bar{r}_o(\bar{v}_o(T_A); t) + R^*(\bar{v}_o(T_A); t) \; \delta\bar{v}(T_A)}$$

$$R^*(t) = \left\| \frac{\partial r_s^{\,i}}{\partial (\cancel{\xi} v(T_A))^j} \right\|$$

Imply only one correction

$$\delta\bar{r} = R(t) \; \delta\bar{v}(T_L)$$

$$\delta\bar{r} = R^*(t) \; \delta\bar{v}(T_A)$$

$$\boxed{\delta\bar{v}(T_A) = R^{*^{-1}}(t) \; R(t) \; \delta\bar{v}(T_L)}$$

$$\bar{v}_s(\bar{v}_s(T_L); t) = \vec{v}_o(v_o(T_L); t) + V(v_o(t_L); t) \; \delta\bar{v}(T_L)$$

$$\bar{v}_s^*(\bar{v}_s(T_A); t) = \vec{v}_o(v_o(T_A); t) + V^*(v_o(T_A); t) \; \delta\bar{v}(T_A)$$

$$V(t) = \left\| \frac{\partial \vec{v}_s^{\,i}}{\partial \cancel{\xi} v_s(T_L)^j} \right\|$$

$$V^*(t) = \left\| \frac{\partial \vec{v}_s^{\,i}}{\partial (\cancel{\xi} v_s(T_A))^j} \right\|$$

Only 4 of 6 matrices are independent.

$$V(t) = C(t)\, R(t)$$

$$V^*(t) = C^*(t)\, R^*(t)$$

$\cancel{A(t) = \frac{\partial r_{sx}}{\partial v_0}}$ — results from partial differentiation

$$\frac{\partial v_{sx}}{\partial(\delta v_x(T_c))} = \frac{\partial v_{sx}}{\partial r_{sx}} \frac{\partial r_{sx}}{\partial(\delta v_x(T_c))} + \cdots$$

Can compute these matrices as partials or from 42 D.E's

Differential Equation approach

① $\frac{dr_s}{dt} = \bar{v}_s$

④ $\frac{d\bar{v}_s}{dt} = -\frac{\mu}{r_s^3}\bar{r}$

Substitute in ① & ② from matrix equation..

$$\frac{d\bar{r}_0}{dt} + \frac{d}{dt}\left[R(t)\, \delta\bar{v}(T_c)\right] = \bar{v}_0 + V(t)\, \delta\bar{v}(T_c)$$

$$\frac{d\bar{v}_0}{dt} + \frac{dV}{dt}\delta\bar{v}(T_c) = -\frac{\mu}{r_s^3}\left(\bar{r}_0 + R\,\delta\bar{v}(T_c)\right)$$

$$\frac{1}{r_s^3} = \frac{1}{\left[(\bar{r}_0+\delta\bar{r})\cdot(\bar{r}_0+\delta\bar{r})\right]^{3/2}} \approx \frac{1}{\left[r_0^2 + 2\bar{r}_0^T\delta\bar{r} + \cdots\right]^{3/2}}$$

$$\approx \frac{1}{r_0^3} - \frac{3\,\bar{r}_0^T\delta\bar{r}}{r_0^5} + \cdots$$

$$\delta \bar{r} = R\, \delta \bar{v}(T_L)$$

$$\frac{1}{r_s^3} = \frac{1}{r_0^3} - \frac{3}{r_0^5} \bar{r}_0^T R\, \delta \bar{v}(T_L)$$

r_0 part of D.E.'s can be subtracted out.

$$\left(\frac{dR}{dt} - V\right) \delta \bar{v}(T_L) = 0$$

$$\left(\frac{dV}{dt} - R_0 R\right) \delta \bar{v}(T_L) = 0$$

$$\boxed{R_0 = \frac{\mu}{r_0^5}\left(3 \bar{r}_0 \bar{r}_0^T - r_0^2 I\right)}$$

Matrix of change of gravity vector wrt. position vector.

$$R_0 = \left\| \frac{\partial \bar{g}}{\partial \bar{r}_s} \right\|$$

$$\boxed{\frac{dR}{dt} = V} \qquad \boxed{\frac{dV}{dt} = R_0 R}$$

Initial Conditions $\boxed{\begin{array}{l} R(T_L) = 0 \\ V(T_L) = I \end{array}}$

Similarly $R^*(T_A) = 0$

$V^*(T_A) = I$ $\leftarrow$ & time must run backward.

18 + 18 + 6 D.E. for reference path = 42 D.E.'s.

20 April 1960

$$R(t) = \left\| \frac{\partial \bar{r}_s}{\partial(\bar{V}_s(T_L))} \right\|$$ Deviation from reference position due to deviation in launch velocity

$$V(t) = \left\| \frac{\partial \bar{V}_s}{\partial(\bar{V}_s(T_L))} \right\|$$ change in the velocity at time t from the reference due to deviation in launch velocity

Same with R^* & V^* treating arrival as launch & flying backward.

$$\frac{dR^*}{dt} = V^*$$ with initial condition $R^*(T_A) = 0$

$R^*(T_A) = I$

$$\frac{dV^*}{dt} = R_0 R^*$$

$C = V R^{-1}$ $\quad C^* = V^* R^{*-1}$ also satisfy D.E.'s

$C^{-1} = R V^{-1}$ $\quad C^{-1}(T_L) = 0$

$$C^{-1} R_0 R + \frac{dC^{-1}}{dt} V = V$$

$$C^{-1} R_0 R V^{-1} + \frac{dC^{-1}}{dt} = I$$

$$\boxed{\frac{dC^{-1}}{dt} + C^{-1} R_0 C^{-1} = I}$$

Note that C is symmetric because R_0 is symmetric

Both satisfy the same D.E. with the same initial conditions.

First ~~and~~ observed empirically & much later did a proof of $C = C^T$

$\delta \underline{r} = \underline{r}_s - \underline{r}_0 \qquad \delta \underline{v} = \underline{v}_s - \underline{v}_0$

$$\boxed{\frac{d(\delta r)}{dt} = \delta \underline{v} \qquad \frac{d(\delta v)}{dt} = R_0 \, \delta \underline{r}}$$ D.E.

Basis of Solutions

$$\boxed{\begin{aligned} \delta \underline{r} &= R \, \underline{c} + R^* \, \underline{c}^* \\ \delta \underline{v} &= V \, \underline{c} + V^* \, \underline{c}^* \end{aligned}}$$

Plug into the D.E., & they satisfy

$\underline{c}$ & $\underline{c}^*$ are arbitrary constants of integration

Example: At $t = t_1$ $\quad \delta \underline{r}(t_1) = \delta \underline{r}_1$

$\delta \underline{v}(t_1) = \delta \underline{v}_1 \qquad R(t_1) = R_1$

R_1^{*-1}

$$\delta \underline{r}_1 = R_1 \underline{c} + R_1^* \underline{c}^* \qquad (1)$$

$$\delta \underline{v}_1 = V_1 \underline{c} + V_1^* \underline{c}^* \qquad (2)$$

$$\underline{c}^* = R_1^{*-1} \delta \underline{r}_1 - R_1^{*-1} R_1 \underline{c} \qquad (3)$$

Substitute (3) in (2)

$$\delta \underline{v}_1 = \left(V_1 - V_1^* R_1^{*-1} R_1 \right) \underline{c} + V_1^* R_1^{*-1} \delta r_1$$

$$\boxed{\begin{aligned} \underline{c} &= \left(V_1 - C_1^* R_1 \right)^{-1} \left(\delta \underline{v}_1 - C_1^* \delta \underline{r}_1 \right) \\ \underline{c}^* &= \left(V_1^* - C_1 R_1^* \right)^{-1} \left(\delta \underline{v}_1 - C_1 \delta r_1 \right) \end{aligned}}$$

Example 2: At t_1: $\delta \underline{r}(t_1) = \delta r_1$

t_2: $\delta \underline{r}(t_2) = \delta r_2$

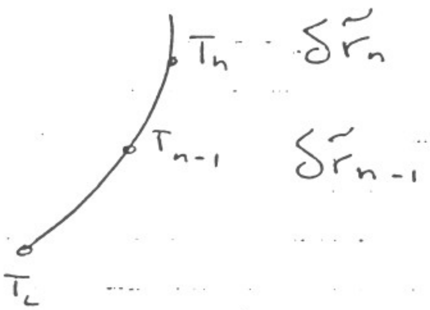
$$\delta \underline{r}_1 = R_1 \underline{c} + R_1^* \underline{c}^*$$

$$\delta \underline{r}_2 = R_2 \underline{c} + R_2^* c^*$$

$$\underline{c} = R_1^{-1} \left(R_2 R_1^{-1} - R_2^* R_1^{*-1} \right)^{-1} \left(\delta \underline{r}_2 - R_2^* R_1^{*-1} \delta r_1 \right)$$

$\underline{c}^*$ = interchange starred & unstarred.

Navigation



before correction

$$\delta \underline{\tilde{v}}_n^- = V_n \underline{c} + V_n^* \underline{c}^*$$

$$\boxed{\delta \underline{\tilde{v}}_n^- = \left(B_n + B_n^* \right) \delta \underline{\tilde{r}}_n + \left(\Gamma_n + \Gamma_n^* \right) \delta \underline{\tilde{r}}_{n-1}}$$

$$A_n = R_{n-1} R_n^{-1} \qquad C_n = V_n R_n^{-1}$$

$$\Gamma_n = C_n \left(A_n - A_n^* \right)^{-1}$$

$$B_n = -\Gamma_n A_n^*$$

$R_n^* \, \delta \underline{v}(T_A) = \delta \underline{\tilde{r}}_n$

$V_n^* \, \delta \underline{v}(T_A) =$ velocity he should have at T_n

$C_n^* \, \delta \underline{\tilde{r}}_n =$ " " " " " "

Velocity correction

$$\boxed{\underline{\tilde{\Delta}}_n = H_n \, \delta \underline{\tilde{r}}_n - P_n \, \delta \underline{\tilde{r}}_{n-1}}$$

$$H_n = C_n^* - (B_n + B_n^*)$$

$$P_n = T_n + T_n^*$$

Fundamental requirement for fixed time of arrival space navigation.

22 April 1960

Fixed Time of Arrival Nav. — FTA

$$\left.\begin{aligned} \delta \underline{r} &= R\underline{c} + R^* \underline{c}^* \\ \delta \underline{v} &= V\underline{c} + V^* \underline{c}^* \end{aligned}\right\}$$ Solution to the D.E.'s

$$\frac{d(\delta \underline{r})}{dt} = \delta \underline{v} \qquad \frac{d \, \delta \underline{v}}{dt} = R_0 \, \delta \underline{r}$$

$$\underline{\tilde{\Delta}}_n = C_n^* \, \delta \underline{\tilde{r}}_n - \delta \underline{\tilde{v}}_n$$

$$= H_n \, \delta \underline{\tilde{r}}_n - P_n \, \delta \underline{\tilde{r}}_{n-1}$$

For a single check point $n = 1$ $\delta \underline{r}(T_L) = 0$

$T_{n-1} = T_L$ $A_1 = 0$ $\Gamma_1 = -C_1 A_1^{*-1}$

$B_1 = -\Gamma_1 A_1^* = C_1$

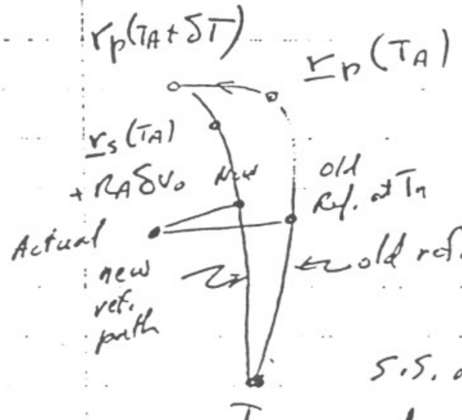
$\tilde{\underline{\Delta}}_1 = H_1 \delta\tilde{r}_1$

$B_1^* = 0$

$\tilde{\underline{\Delta}}_1 = \left(C_1^* - C_1\right) \delta\tilde{\underline{r}}_1$ CHECKS

Variable Time of Arrival Navigation   VTA

Each correction gives a new ETA



$\delta \underline{v}_0(T_L)$ = vel. change at launch to establish the new trajectory

$R_A = R(T_A)$

S.S. at T_A: along new ref. path. $\underline{r}_s(T_A) + R_A\,\delta\underline{v}_0(T_L)$

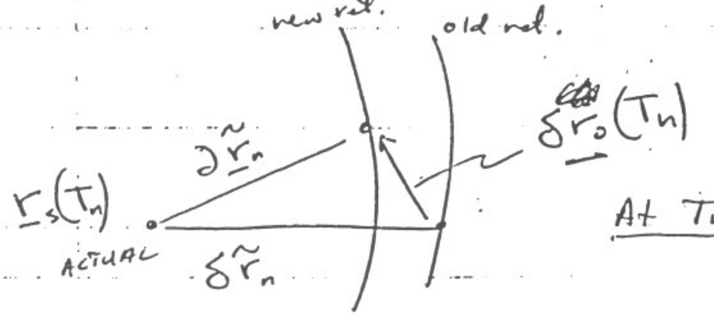
S.S. at $T_A + \delta T$: along new ref. path. $\underline{r}_s(T_A) + R_A\,\delta\underline{v}_0(T_L) + \underline{v}_s(T_A)\,\delta T$

Planet at $T_A + \delta T = \underline{r}_p(T_A) + \underline{v}_p(T_A)\,\delta T$

Determine $\delta\underline{v}_0(T_L)$ so that SS & planet coincide

$$\boxed{\delta\underline{v}_0(T_L) = -R_A^{-1}\,\underline{v}_R(T_A)\,\delta T}$$

$\underline{v}_R(T_A) = \underline{v}_s(T_A) - \underline{v}_p(T_A)$ = relative velocity between the S.S. & planet at time of arrival



At Time T_n

$$\delta \underline{r}_0 (T_n) = R_n \, \delta \underline{v}_0 (T_L)$$

$$\delta \underline{v}_0 (T_n) = V_n \, \delta \underline{v}_0 (T_L)$$

$\tilde{\underline{\Delta}}'_n$ = velocity req'd to go to new rendezvous pt.

$$\tilde{\underline{\Delta}}'_n = C_n^* \, \partial \tilde{\underline{r}}_n - \partial \tilde{\underline{v}}_n^- \qquad (1)$$

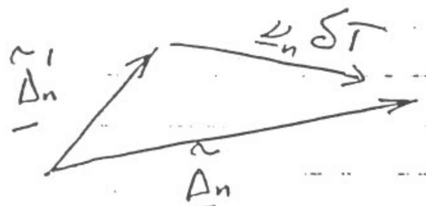
$$\partial \tilde{\underline{r}}_n = \delta \tilde{\underline{r}}_n - \delta \underline{r}_0 (T_n)$$

$$\partial \tilde{\underline{v}}_n^- = \delta \tilde{\underline{v}}_n^- - \delta \underline{v}_0 (T_n)$$

Substitute in (1) & algebra.

$$\boxed{\begin{aligned} \tilde{\underline{\Delta}}'_n &= \tilde{\underline{\Delta}}_n - \underline{\nu}_n \, \delta T \\ \underline{\nu}_n &= (V_n - C_n^* R_n) R_A^{-1} \underline{\nu}_R (T_A) \end{aligned}}$$

Parameter δT is still at our disposal



choose δT to minimize $\tilde{\underline{\Delta}}'_n$

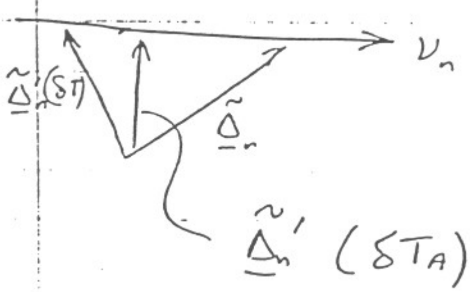
i.e. $\tilde{\underline{\Delta}}'_n \perp \underline{\nu}_n$

Def $\delta T \triangleq \delta T_A$

$$\delta T_A = \frac{\underline{\tilde{\Delta}}_n \cdot \underline{\nu}_n}{\underline{\nu}_n \cdot \underline{\nu}_n}$$

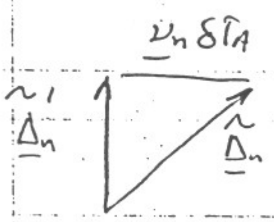
25 April 1960 VTA Nav.

$$\underline{\tilde{\Delta}}'_n = \underline{\tilde{\Delta}}_n - \underline{\nu}_n \, \delta T \quad \{= f(\delta T) \text{ note}\}$$



$$\underline{\nu}_n \, \delta T_A = \frac{(\underline{\nu}_n \underline{\nu}_n^T)}{\underline{\nu}_n \cdot \underline{\nu}_n} \underline{\tilde{\Delta}}_n \qquad \underline{\nu}_n \underline{\nu}_n^T = 3 \times 3 \text{ matrix}$$

$$\underline{\tilde{\Delta}}'_n(\delta T_A) = \left(I - \frac{\nu_n \nu_n^T}{\underline{\nu}_n \cdot \underline{\nu}_n}\right) \underline{\tilde{\Delta}}_n = M_n \underline{\tilde{\Delta}}_n$$



M matrix projects $\underline{\tilde{\Delta}}_n$ onto a direction normal to $\nu_n \, \delta T_A$

M_n^{-1} does not exist.

$$\left| I - \frac{\nu_n \cdot \nu_n^T}{\underline{\nu}_n \cdot \underline{\nu}_n} \right| = 0$$

Try to show this.

Can't get a vector from one of its components.

Get velocity errors due to launch velocity error, imperfect measurement, imperfect corrections.

Error analysis

$$\delta \underline{\tilde{r}}_n = \delta \underline{r}_n + \underline{\epsilon}_n$$

$$\delta \underline{\tilde{v}}_n^- = \delta \underline{v}_n^- + \underline{\delta}_n$$

$$\underline{\tilde{\Delta}}_n = \underline{\Delta}_n + \underline{\eta}_n \qquad \eta_n \text{ is error in applied velocity}$$

Let $n=0 \Rightarrow T_L = T_0$

$$\underline{\Delta}_0 = -\underline{\eta}_0 \qquad \text{Error in Launch Velocity}$$

$$\underline{\tilde{\Delta}}_n' = M_n\left(C_n^* \,\delta \underline{\tilde{r}}_n - \delta \underline{\tilde{v}}_n^-\right)$$

$$\underline{\Delta}_n' = M_n\left(C_n^* \,\delta \underline{r}_n - \delta \underline{v}_n^- + C_n^* \underline{\epsilon}_n - \underline{\delta}_n\right) - \underline{\eta}_n$$

$$\underline{\Delta}_n' = M_n\left(C_n^* \,\delta \underline{r}_n - \delta v_n^-\right) + M_n\left(H_n \epsilon_n - P_n \underline{\epsilon}_{n-1}\right) - \underline{\eta}_n$$

"Fancy footwork now"

$$C_n^* : \quad \delta r_n = R_n \underline{c} + R_n^* c^*$$

$$-I : \quad \delta \underline{v}_n^- = V_n \underline{c} + V_n^* c^*$$

Add together

$$C_n^* \delta \underline{r}_n - \delta v_n^- = \left(C_n^* R_n - V_n\right) \underline{c} + \cancel{0}$$

$$= -\Lambda_n \, c \qquad \Lambda_n = V_n - C_n^* R_n$$

$$\underline{c} = -\Lambda_n^{-1}\left(C_n^* \delta \underline{r}_n - \delta \underline{v}_n^-\right)$$

Similarly $\quad C_{n-1}^* \delta \underline{r}_{n-1} - \delta \underline{v}_{n-1}^+ = -\Lambda_{n-1}\, \underline{c}$

$$= \Lambda_{n-1} \Lambda_n^{-1}\left(C_n^* \delta \underline{r}_n - \delta \underline{v}_n^-\right)$$

Recursion Relation

$$\boxed{\left(C_{n-1}^* \delta \underline{r}_{n-1} - \delta \underline{v}_{n-1}^-\right) - \underline{\Delta}_{n-1} = \Lambda_{n-1} \Lambda_n^{-1}\left(C_n^* \delta \underline{r}_n - \delta \underline{v}_n^{-1}\right)}$$

$M_n \stackrel{\Delta}{=} I$ for fixed time of arrival $\therefore \; \frac{1}{\underline{v}_n \cdot \underline{v}_n}$ undefined for $\underline{v}_n =$

$$\boxed{\begin{aligned}\underline{\Delta}_n = H_n \underline{\epsilon}_n - \left(P_n + \Lambda_n \Lambda_{n-1}^{-1} H_{n-1}\right) \underline{\epsilon}_{n-1} \\ + \Lambda_n \Lambda_{n-1}^{-1} P_{n-1} \underline{\epsilon}_{n-2} = \eta_n + \Lambda_n \Lambda_{n-1}^{-1} \eta_{n-1}\end{aligned}}$$

27 April 1960   thus Exam last class - ½ Rel. ½ Nav.

For variable time of arrival we will use recursion relation

$$C_n^* \delta \underline{r}_n - \delta \underline{v}_n^- = -\Lambda_n \sum_{k=0}^{n-1} \Lambda_k^{-1} \underline{\Delta}_k'$$

$$\underline{\Delta}_n' = M_n\left(H_n \underline{\epsilon}_n - P_n \epsilon_{n-1}\right) - \eta_n - M_n \Lambda_n \sum_{k=0}^{n-1} \Lambda_k^{-1} \underline{\Delta}_k'$$

Would like to show that this only depends on last 2 corrections.

By long calculation:

$$\underline{\Delta}_n' = M_n(H_n \underline{\epsilon}_n - P_n \underline{\epsilon}_{n-1}) - \eta_n$$
$$- M_n \Lambda_n \sum_{k=1}^{n-1} X_{k,n} \Lambda_k^{-1} \left[M_k (H_k \underline{\epsilon}_k - P_k \underline{\epsilon}_{k-1}) - \eta_k \right]$$

where: $X_{k,n} \begin{cases} = I & \text{for } k = n-1 \\ = \prod_{j=n-1}^{k+1} (I - \Lambda_j^{-1} M_j \Lambda_j) & \text{for } k \leq n-2 \end{cases}$

All terms for $k \leq (n-2)$ have as factor

$$M_n \Lambda_n (I - \Lambda_{n-1}^{-1} M_{n-1} \Lambda_{n-1})$$

Prob is to show the factor $= 0$

Then

$$\underline{\Delta}_n' = M_n H_n \underline{\epsilon}_n - (M_n P_n + M_n \Lambda_n \Lambda_{n-1}^{-1} M_{n-1} H_{n-1}) \underline{\epsilon}_{n-1}$$
$$+ M_n \Lambda_n \Lambda_{n-1}^{-1} M_{n-1} P_{n-1} \underline{\epsilon}_{n-2} \quad - \eta_n$$
$$+ M_n \Lambda_n \Lambda_{n-1}^{-1} \eta_{n-1}$$

which reduces to FTA for $M = I$

Optimum Time for Fixes.

~~Weighting~~ — Tr matrix

$$\epsilon_n = (CF)\,\underline{d}$$

$E_n = \overline{\epsilon_n \epsilon_n^T}$ correlation matrix of $\underline{\epsilon}_n$

RMS uncertainty $= \left[\overline{\text{Trace}\{E_n\}}\right]^{1/2}$

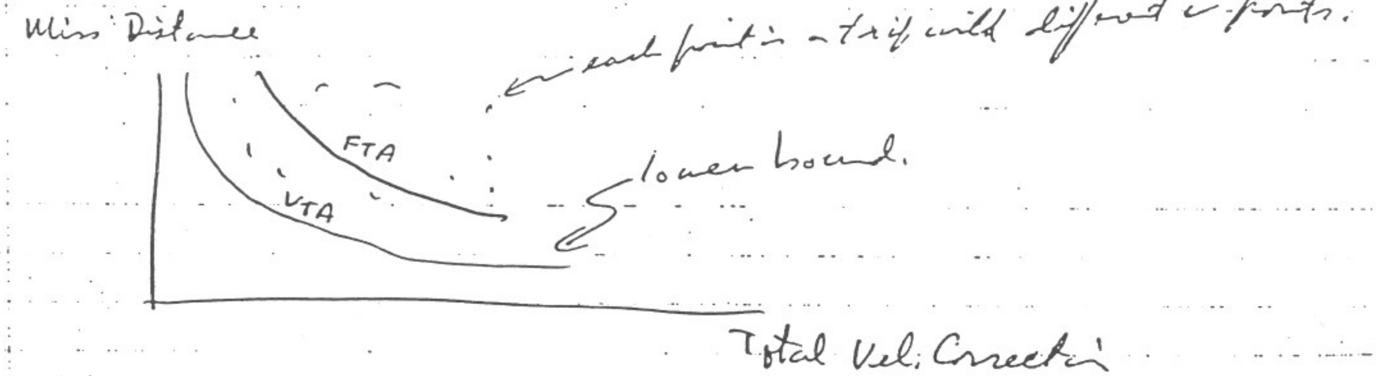
Find Trace $\left\{\overline{\underline{\Delta}_n' \underline{\Delta}_n'^T}\right\}$ = mean squared velocity correction at T_n

Take root of each of these & add them up to get the 1σ amount of fuel required.

Good idea to carry 3σ fuel supply

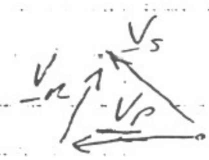
What is the optimum point to take fix? Virgin Field

Computer Monte Carlo optimum program game:



Miss distance 20 mi OR 100 ft/sec by Computer.

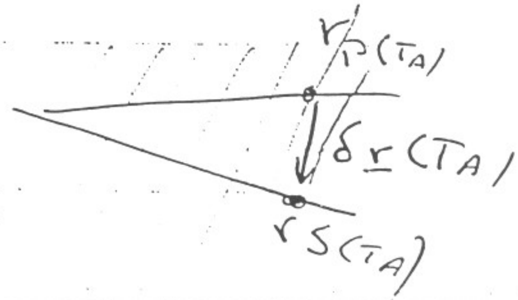
Miss Distance



$$\delta\underline{r}(T_A) = R_A \sum_{k=0}^{N} \Lambda_k^{-1}\,\underline{\Delta}_k'$$

N = no. of check points

Should only consider component $\perp$ to relative motion



29 April 1960

MONDAY 35-332 10 AM

excerpt:- ATWOOD BLASKE EDMONDS
MILLON SMITH STERN SELF

R-273 for VTA - Secret

What happens at the planet

$$\underline{\Delta}_n' = M_n\left(C_n^* \delta \underline{r}_n - \delta \underline{v}_n^-\right) + M_n\left(H_n \underline{\epsilon}_n - P_n \underline{\epsilon}_{n-1}\right) - \underline{\eta}_n$$

$$C_n^* \delta r_n - \delta \underline{v}_n^- = -\Lambda_n \sum_{k=0}^{n-1} \Lambda_k^{-1} \underline{\Delta}_k'$$

at last check point:

$$-C_N^*: \qquad \delta \underline{r}_N = R_N \underline{c} + R_N^* \underline{c}^* \qquad (1)$$

$$\delta \underline{v}_N^+ = V_N \underline{c} + V_N^* \underline{c}^* \qquad (2)$$

$$\delta \underline{r}(T_A) = R_A \underline{c} + \cancel{R_A^* \underline{c}^*}$$

Adding (1) & (2) $\qquad \delta \underline{v}_N^+ - C_N \delta r_N = \Lambda_n \underline{c}$

$$\delta \underline{r}(T_A) = R_A \Lambda_N^{-1}\left(\delta \underline{v}_n^- + \underline{\Delta}_n - C_n^* \delta \underline{r}_N\right)$$

$$\boxed{\delta \underline{r}(T_A) = R_A \sum_{k=0}^{N} \Lambda_k^{-1} \underline{\Delta}_k'}$$

Now get the component of $\delta \underline{r}(T_A)$ normal to $\underline{V}_R(T_A)$ - relative to planet

$$\delta \underline{r}_a = M_a \, \delta \underline{r}(T_A)$$

$$M_a = I - \frac{\underline{v}_R^*(T_A)\, \underline{v}_R^T(T_A)}{\underline{v}_R(T_A) \cdot \underline{v}_R(T_A)}$$

$$\delta \underline{r}(T_A) = R_A \Lambda_N^{-1} M_N \left(H_N \underline{\epsilon}_N - P_N \underline{\epsilon}_{N-1} \right) - R_A \Delta_N^{-1} \underline{\eta}_N$$

$$+ R_A \left(I - \Lambda_N^{-1} M_N \Lambda_N \right) \sum_{k=0}^{N-1} \Lambda_k^{-1} \underline{\Delta}_k'$$

Multiplication by M_a annihilates the last term.

$$M_N = I - \frac{\underline{\nu}_N \underline{\nu}_N^T}{\underline{\nu}_N \cdot \underline{\nu}_N} \qquad \underline{\nu}_N = \Lambda_N R_A^{-1} \underline{v}_R(T_A)$$

Have to show $\quad M_a R_A \Lambda_N^{-1} \underline{\nu}_N \underline{\nu}_N^T \Lambda_N / (\underline{\nu}_N \cdot \underline{\nu}_N) \stackrel{?}{=} 0$

$$M_a(I - M_a) = 0 \qquad \text{Try This}$$

$$\underline{\delta \underline{r}_a = M_a R_A \Lambda_N^{-1} M_N \left(H_N \epsilon_N - P_N \underline{\epsilon}_{N-1} \right)} - M_a R_A \Lambda_N^{-1} \underline{\eta}_N$$

for FTA $M_N = I$ & still valid.

<u>Velocity Error at Destination</u>

$$0 = R_n \underline{c} + R_n^* \underline{c}^*$$

$$\underline{\Delta}_n' = V_n \underline{c} + V_n^* \underline{c}^*$$

Considering one Δ_n' & will then $\sum$ by linearity

$$\underline{c} = \Lambda_n^{-1} \Delta_n' \qquad \underline{c}^* = \Lambda_n^{*-1} \underline{\Delta}_n'$$

$$\delta \underline{V}(T_A) = \sum_{n=0}^{N} \left(V_A \Lambda_n^{-1} + \Lambda_n^{*-1} \right) \underline{\Delta}_n'$$

Factor out Λ_n^{-1}

$$\boxed{\delta V(T_A) = \sum_{n=0}^{N} \left(V_A - R_A^{*-1} R_n \right) \Lambda_n^{-1} \underline{\Delta}_n'}$$

Change in Arrival Time

$$\delta\tilde{T} = \frac{\tilde{\underline{\Delta}}_n \cdot \underline{\nu}_n}{\underline{\nu}_n \cdot \underline{V}_n}$$

Only one that counts is

$$\frac{\tilde{\underline{\Delta}}_N \cdot \underline{\nu}_N}{\underline{\nu}_N \cdot \underline{V}_N}$$

Substitute for $\tilde{\underline{\Delta}}_N$

$$\rightarrow \delta\tilde{T}_A = (\underline{\nu}_N \cdot \underline{V}_N)^{-1} \left\{ \nu_N^T \left[H_N \underline{\epsilon}_N - P_N \underline{\epsilon}_{N-1} - \Lambda_N \sum_{k=0}^{N-1} \Lambda_k^{-1} \underline{\Delta}_k' \right] \right\}$$

2 May 1960 — Relativity

P.G. Bergmann "Introduction to Theory of Relativity" — Prentice-Hall

Michelson-Morley interferometer exp. - know for doctor's exams.

Philosophy of using a theory in area where it works & where the sophistication is needed

Read Introduction to Bergmann

Pierce - "Clock Paradox" - J.J. Coupling

IRE P. 47 Je '59 1053-61 Relativity & Space Travel.

Read Chap III & Chap IV → Lorentz Transformation p. 38

4 May 1960

$t = \frac{d}{c}$ $\quad \frac{\Delta t}{\Delta c} = -\frac{d}{c^2}$

$v = 10^6$ cm/sec $\quad d = 10^{18}$ cm

Double Stars. ← Throws out corpuscular theory.

$\Delta t > 10^3$ sec

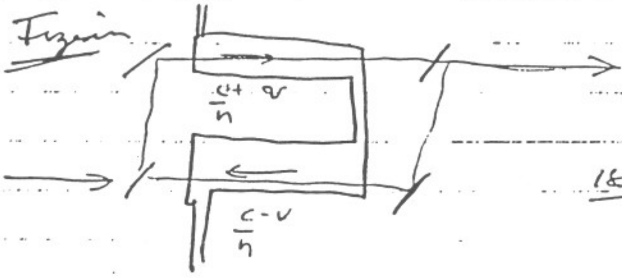
Never see stars in hodgepodge. Always eclipse each other regularly.

Can actually see star in 2 different places at same time. Try this.

Work out
p.45 Prob #3

Doppler
Freq. Shift. Classical vs. Relativistic
Pierce uses this

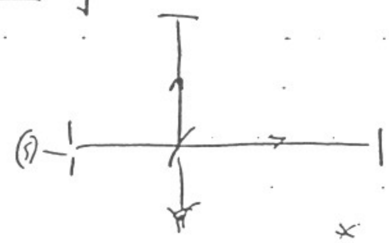
Fizeau



$$u = \frac{c}{n} \pm v\left(1 - \frac{1}{n^2}\right)$$

1850 $n = 1.33$ for H_2O

Michelson & Morley



Bradley

41" sec — M/M looked like complete ellipse drag.

Next time :- Lorentz. - reconciled Maxwell's Equation

Home Prob. 1) $c = c'$
2) Laws of Physics the same in all inertial frames.

Gallilean $x' = x - vt$ $\quad x' = \alpha(x - vt)$ Possibility
$x = x' + vt$ $\quad x = \alpha(x' + vt)$

Lorentz $\begin{cases} x' = \alpha(x - vt) \\ x = \alpha(x' + vt') \end{cases}$

Do for Friday.

See RK & Lorentz

Lindsay & Margenau —
Foundations of Physics
Good Book — Dover Edition.

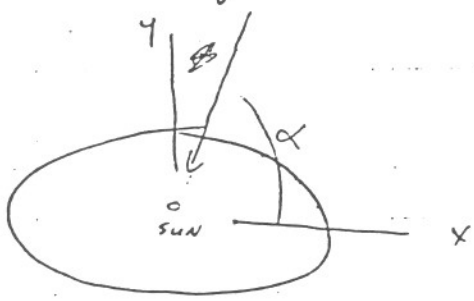
6 May 1960

Monday - Hand in Prob. on Transformation
Read rest of Chap
& Do Prob #3 p.45
Do lengthat 2 times.

$$x' = \frac{x - vt}{\sqrt{1-\beta^2}} \qquad ct' = \frac{ct - \beta x}{\sqrt{1-\beta^2}}$$

$$x = \frac{x' + vt'}{\sqrt{1-\beta^2}} \qquad ct = \frac{ct' + \beta x'}{\sqrt{1-\beta^2}} \quad -\ -$$

Explanation of Abberation



$$\left.\begin{array}{l} x = ct \cos\alpha \\ y = ct \sin\alpha \end{array}\right\} \text{ point on wave front in helio centric plane.}$$

express coordinates of point on wave front in earth centered frame.

Transforming $\quad x' + vt' = c\left(t' + \frac{v}{c^2}x'\right)\cos\alpha$

$$y'\sqrt{1-\frac{v^2}{c^2}} = c\left(t' + \frac{v}{c^2}x'\right)\sin\alpha$$

$$\tan\alpha^* = \frac{y^*}{x^*} \qquad \operatorname{ctn}\alpha^* = \frac{\operatorname{ctn}\alpha - \frac{v}{c}\csc\alpha}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\doteq \operatorname{ctn}\alpha - \frac{v}{c}\csc\alpha \quad ---$$

Fizeau's Exp. $\quad$ x = coord. of wavefront $\quad$ take one frame in H_2O

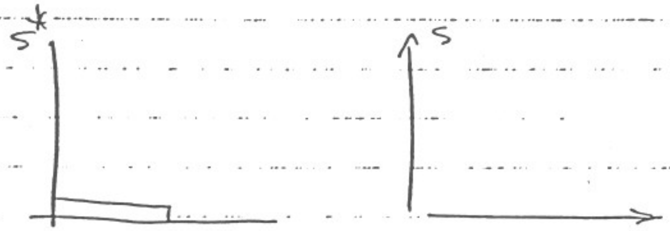
$$x - vt = \frac{c}{n}\left(t - \frac{v}{c^2}x\right)$$

$$x = \frac{\frac{c}{n} + v}{1 + \frac{v}{nc}}\,t \qquad \frac{x}{t} = \frac{c}{n} + \frac{1 - \frac{1}{n^2}}{1 + \frac{v}{nc}}\,v = \text{velocity of wave front in } H_2O$$

Philosophy of Relativity :-

Rod does not physically get short. But any measurement of its length does get less.

Box at rest in starred frame.



$x_1^* = 0$

$x_2^* = L$

$x_2^* - x_1^* = L^* = L_0$

Find these coordinates in S frame

$$x_1^* = x_1 - v t_1 \; 1/\sqrt{1-\beta^2}$$

$$x_2^* = \frac{x_2 - v t_2}{\sqrt{1-\beta^2}}$$

$L_0 = x_1^* - x_2^*$

$$L_0 = \frac{L_i - v(t_2 - t_1)}{\sqrt{1-\beta^2}}$$

$L_i = x_2 - x_1$ = indicated length $= \sqrt{1-\beta^2}\, L_0 + v(t_2 - t_1)$

Can synchronize clocks in any one frame $\longrightarrow$ vanishes if we compare ends of box at the same time in S

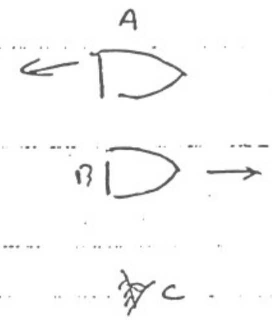
Do this looking at 2 ends of rod as they go by.

9 May 1960

Energy cannot go faster than c
Information " " " " c

Phase velocity may exceed c (mathematically.)

Pierce

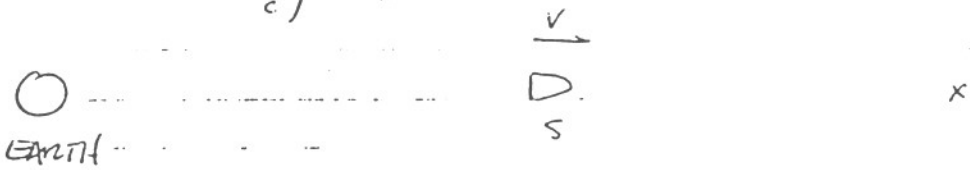


Lifting Bolts not simultaneous.

Peter Pan Paradox

$$f_i = f_0\left(\frac{1-\frac{v}{c}}{1+\frac{v}{c}}\right)$$

Measure time by counting pulses.



Earth & ship emit f_0 radiation as they calibrate it in their own frame.

$$t_E = \frac{2L}{v}$$

$$n_{EE} = f_0 t_E = \frac{2L}{v} f_0$$

earth ~~generated~~ measured; earth ~~measured~~ generated.

$$f_{ES} = f_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

$t = \frac{L}{c}\,\frac{L}{v} = \frac{L}{c}\left(1+\frac{v}{c}\right)$ outbound

$t = \frac{L}{v} - \frac{L}{c}$ inbound. receiving time.

$$n_{ES} = f_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\,\frac{L}{v}\left(1+\frac{v}{c}\right) + f_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\,\frac{L}{v}\left(1-\frac{v}{c}\right)$$

$$= f_0\frac{L}{v}\,2\sqrt{1-\frac{v^2}{c^2}}$$

Now look at $n_{SHIP} = \frac{2L\sqrt{1-\frac{v^2}{c^2}}}{v} f_0$

They agree on the counted number of pulses that ship sent

$f_{SE} = f_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$ outbound $f_{SE} = f_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$ inbound

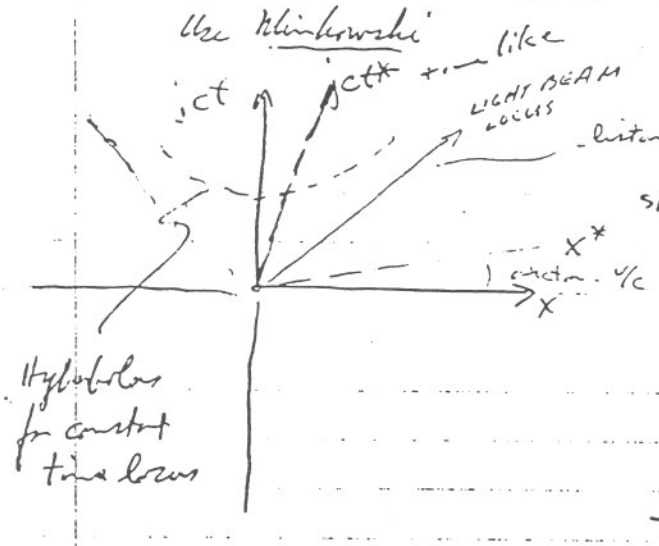
$t_{out} = \frac{L\sqrt{1-\frac{v^2}{c^2}}}{v}$ $t_{in} = \frac{L\sqrt{1-\frac{v^2}{c^2}}}{v}$

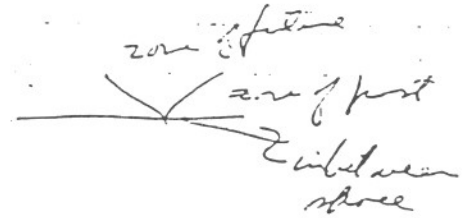
$n_{SE} = \frac{2L}{v} f_0$ if you multiply out. They agree on the number of pulses earth sent.

The ship was the one that accelerated.

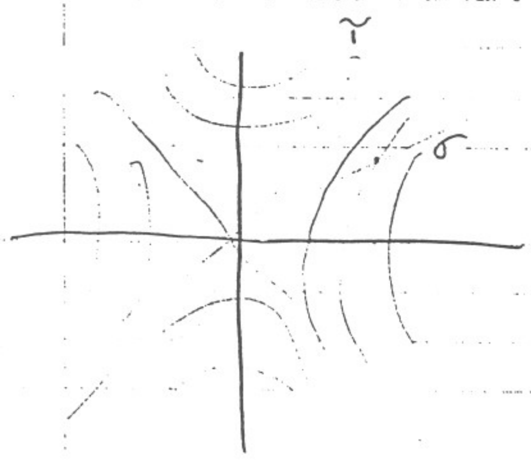
<u>No class Wednesday</u> - Read Chap. 6. next. $\Rightarrow$ p. 96.

<u>13 May ACU</u>





$\Rightarrow$ For Monday Treat Pixie's Event on Minkowski Plot. Qualitatively (Semi Quantitative,



Read <u>Yellow Book</u> on this & <u>Lindsay & Margenau</u>

16 May 1960

Wed Chap IX p.137 Read.
Chap X Prin. of Equivalence.
p.99 Influence of Gravitation on Prop. of Light & Gen Relativity
-p.120
by A. Einstein Yellow Book.

p. 345 Lindsay & Margenau

$\bar{F} = \frac{d}{dt}(m\bar{v})$ $\quad \bar{p} = m\bar{v}$ Classically

$\bar{Q}$ = 4 vector of ~~mass~~ velocity $\quad ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$

$\bar{Q} = \frac{d\bar{s}}{d\tau}$ $\quad \tau$ = proper time

$Q_x = \frac{ds}{dt}\frac{dt}{d\tau} = v_x / \sqrt{1-\beta^2}$

$Q_t = jct / \sqrt{1-\beta^2}$

$\bar{p} = m_0 \frac{d\bar{s}}{d\tau}$ $\quad F = \frac{d}{d\tau}(m_0 \bar{Q}) = \frac{d\bar{p}}{d\tau}$

β not constant with forces.

$$\bar{F}\sqrt{1-\beta^2} = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1-\beta^2}}\right) = m_0 \bar{v}\frac{d}{dt}\left(\frac{1}{\sqrt{1-\beta^2}}\right) + \frac{m_0}{\sqrt{1-\beta^2}}\frac{d\bar{v}}{dt}$$

force & acceleration are not parallel in 4 space

⇒ Operations Research Book

18 May 1960 $\quad E^2 - H^2$ = invariant.

$E_x' = E_x \qquad H_x' = H_x$

$E_y' = \frac{1}{\sqrt{1-\beta^2}}(E_y - \beta H_z) \qquad H_y' = \frac{1}{\sqrt{1-\beta^2}}(H_y + \beta E_z)$

$E_z' = \frac{1}{\sqrt{1-\beta^2}}(E_z + \beta H_y) \qquad H_z' = \frac{1}{\sqrt{1-\beta^2}}(H_z - \beta E_y)$

Prin. of Equivalence Inertial Mass = Gravitational Mass.

QUIZ How CLOSED BOOK

20 May 1960

Bridgman - Harvard - doesn't agree with General Relativity

Struik's Differential Geometry